Singularity resolution in fuzzy de Sitter cosmology*

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Abstract

We review and discuss properties of the recently introduced fuzzy de Sitter space [1, 2, 3], in particular its predictions in cosmology.

1. Introduction

One of pending problems in theoretical physics is description of the structure of spacetime at small distances, that is (if we believe in Einstein's insight on the relation between geometry and gravity), quantization of gravity. Since straightforward quantization of the gravitational field does not work, other ideas are developed. On the line of physics, it is for example the idea that there is an elementary substructure (like strings) exists, that it can be quantized by the usual methods yielding the 'quanta of spacetime'. On the line of mathematics, the idea that there is yet to be discovered algebraic structure that generalizes the existing notion of geometry, and provides Einstein's gravity in the large scale limit. There are many ideas in between, combining quantum field theory with geometry. Quantum spacetime should, or should have a potential to, solve two main problems of the current description of space, time and matter: singularities in general relativity and divergences in quantum field theory.

We usually belive that spacetime at small scales will have some kind of geometric structure, different from that of a manifold: perhaps discrete. This structure might be effective or emergent, but also fundamental. A discrete structure physicists are very familiar with is that of an algebra, e.g. the algebra of operators in Hilbert space or a Lie algebra. In noncommutative geometry one assumes that spacetime is described by an algebra of operators, more precisely, a C^* algebra. Furthermore, in order to describe

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fields and equations of motion, one introduces derivatives. In principle, the goal is to introduce noncommutative differential geometry (as quantum spacetime), and describe classical and quantized fields on it. There are several different approaches to this: we work in the following within the noncommutative frame formalism of Madore, [4].

A paradigmatic example of a noncommutative space is the fuzzy sphere, [5]. To construct it one uses its symmetry: here we use similar construction, aiming to obtain four-dimensional fuzzy spacetimes with spherical symmetry. This will enable, we believe, to define noncommutative generalizations of important configurations such as black holes or cosmological spacetimes. As the initial step, we define four-dimensional fuzzy de Sitter space, using the algebra of the de Sitter group SO(1,4) and its unitary irreducible representations.

2. Noncommutative frames

A noncommutative space is an algebra $\mathcal{A}$ generated by a set of hermitian coordinates x^μ

$$[x^\mu, x^\nu] = i\bar{k} J^{\mu\nu}(x). \tag{1}$$

We can either have an abstract position algebra or its concrete representation, ideally we discuss both. The structure of a noncommutative space can for example be described by the spectra of its coordinates x^μ. There are however other ways to characterize a fuzzy space like its symmetries, or a set of coherent states, or very importantly, its commutative/large scale limit. Diffeomorphisms of noncommutative space are functions on the algebra. Obviously, changes of coordinates change their spectrum. Therefore, although it became quite usual to relate the term 'fuzzy' with either discrete spectrum or finite-dimensional representations, we will in the following identify 'fuzzy' with noncommutative, not presuming *a priori* any properties of the spectra.

Differential structure of $\mathcal{A}$ is given by the momentum algebra. Momenta p_α define a set of vector fields e_α – the free falling frame or tetrad – by

$$e_\alpha f = [p_\alpha, f]. \tag{2}$$

Commutator satisfies the Leibniz rule, and therefore e_α are indeed derivations. In many examples of fuzzy spaces one can identify the algebra generated by momenta with the algebra generated by coordinates, and thus assume that $\mathcal{A}$ is given by either of the sets. But as we want to include, as a particular case, the usual commutative geometry, we do not do so. On commutative manifold, moving frame is given by $e_\alpha f = e_\alpha^\mu (\partial_\mu f)$, that is

$$p_\alpha = e_\alpha^\mu \partial_\mu, \quad e_\alpha^\mu = [p_\alpha, x^\mu]\,. \tag{3}$$

Momenta lie outside the coordinate algebra $\mathcal{A}$, e_α are outer derivatives. The space of vector fields, tangent space, has dimension equal to dimension of spacetime.

In the noncommutative case we can define, in analogy

$$e^\mu_\alpha = [p_\alpha, x^\mu], \qquad g^{\mu\nu} = e^\mu_\alpha\, e^\nu_\beta\, \eta^{\alpha\beta}\,. \tag{4}$$

Geometry is defined by the choice of a set of momenta, which now does not have *a priori* fixed number of elements, as the tangent space of a noncommutative algebra is infinite-dimensional. One can also define the frame 1-forms θ^α dual to e_α, and the differential d

$$df = (e_\alpha f)\,\theta^\alpha, \qquad [\theta^\alpha, \theta^\beta] = 0. \tag{5}$$

There are additional conditions [4], to assure orthonormality of the moving frame and compatibility of the algebraic (1) with the differential structure (2). If we work with an abstract algebra, the Jacobi identities are also imposed as constraints. It is then possible to define differential-geometric quantities like connection, covariant derivative, curvature and torsion, by formulas analogous to those given in the Cartan's description of geometry. Laplacian of a scalar function is defined naturally,

$$\Delta f = \eta^{\alpha\beta}[p_\alpha, [p_\beta, f]]. \tag{6}$$

Thus one can describe curved noncommutative spaces as well as scalar, spinor and gauge fields. One notion that is representation-dependent is that of a trace, and therefore on an abstract noncommutative space the action is defined only formally.

3. Fuzzy de Sitter space

Construction which we describe here shows that it is in fact possible to adapt and generalize the fuzzy sphere, and obtain noncommutative extensions for all homogeneous spaces. We discuss four-dimensional fuzzy de Sitter space. In the commutative case, de Sitter space is defined as the embedding, [6]

$$-v^2 + w^2 + x^2 + y^2 + z^2 = \frac{3}{\Lambda} \tag{7}$$

in the flat five-dimensional space

$$ds^2 = -dv^2 + dw^2 + dx^2 + dy^2 + dz^2. \tag{8}$$

It has maximal symmetry. We (and originally, [7, 8]) define fuzzy de Sitter space using the algebra of its symmetry group SO(1,4). It has ten generators $M_{\alpha\beta}$,

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -i(\eta_{\alpha\gamma}M_{\beta\delta} - \eta_{\alpha\delta}M_{\beta\gamma} - \eta_{\beta\gamma}M_{\alpha\delta} + \eta_{\beta\delta}M_{\alpha\gamma}) \tag{9}$$

$\alpha, \beta, ... = 0, 1, 2, 3, 4$; we use signature $\eta_{\alpha\beta} = \mathrm{diag}(+ - - - -)$. The SO(1,4) has two Casimir operators, quadratic and quartic

$$\mathcal{Q} = -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta}, \tag{10}$$

$$\mathcal{W} = -W_\alpha W^\alpha, \quad W_\alpha = \frac{1}{8} \epsilon_{\alpha\beta\gamma\delta\eta} M^{\beta\gamma} M^{\delta\eta}. \tag{11}$$

Clearly, relation $\mathcal{W}$=const would be analogous to the embedding of (7) of the four-dimensional commutative de Sitter space in five flat dimensions. Therefore, we introduce coordinates as

$$x^\alpha = \ell \, W^\alpha \tag{12}$$

and define fuzzy de Sitter space to be a unitary irreducible representation (UIR) of the de Sitter algebra. The quartic Casimir of SO(1,4) is then related to the cosmological constant, as $\eta_{\alpha\beta} \, x^\alpha x^\beta = \ell^2 \, W^\alpha W_\alpha = 3/\Lambda$.

Coordinates x^α are quadratic in the group generators, and in general they do not close into a Lie or quadratic algebra under commutation:

$$[W^\alpha, W^\beta] = -\frac{i}{2} \epsilon^{\alpha\beta\gamma\delta\eta} \, W_\gamma M_{\delta\eta} \, . \tag{13}$$

However, one can show [2], that in irreducible representations they generate the whole algebra via

$$i \, \mathcal{W} M^{\rho\sigma} = [W^\rho, W^\sigma] + \frac{1}{2} \epsilon^{\alpha\mu\rho\sigma\tau} W_\tau [W_\alpha, W_\mu]. \tag{14}$$

We will see later that this formula is the Fourier transformation.

We have (at least) two choices of momenta that give de Sitter metric in the commutative limit of this fuzzy space, [1]. The simplest is

$$ip_0 = M_{04} \, , \qquad ip_i = M_{i4} + M_{0i}. \tag{15}$$

The frame formalism gives us the line element,

$$ds^2 = d\tau^2 - e^{-2\tau} \, (dx^i)^2 \tag{16}$$

where cosmic time τ is defined by

$$\frac{\tau}{\ell} = \log(W_0 - W_4) \, . \tag{17}$$

In the conformal group notation, $M_{i4} + M_{0i}$ are translations and M_{04} is dilatation. From

$$[\, iM_{04}, W_0 - W_4] = W_0 - W_4 \tag{18}$$

we find that dilatation is canonically conjugate to the cosmic time, i.e, it can be identified with the hamiltonian.

Unitary irreducible representations of de Sitter group are known, found in [9]. They are labelled by two quantum numbers (s, ρ) or (ν, q) and fall into following categories:

- Principal continuous series: $\rho \geq 0$, $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$
  $\mathcal{Q} = -s(s+1) + \frac{9}{4} + \rho^2, \quad \mathcal{W} = s(s+1)(\frac{1}{4} + \rho^2)$
- Complementary continuous series: $\nu \in R$, $|\nu| < \frac{3}{2}$, $s = 0, 1, 2 \dots$
  $\mathcal{Q} = -s(s+1) + \frac{9}{4} - \nu^2, \quad \mathcal{W} = s(s+1)(\frac{1}{4} - \nu^2)$
- Discrete series: $s = \frac{1}{2}, 1, \frac{3}{2}, 2 \dots$, $q = s, s-1, \dots 0$ or $\frac{1}{2}$
  $\mathcal{Q} = -s(s+1) - (q+1)(q-2), \quad \mathcal{W} = s(s+1)q(q-1)$.

4. Spatial coordinates

The principal continuous series has Hilbert space representations [10], which are given in the Bargmann-Wigner representation spaces [11] of the unitary irreducible representations of the Poincaré group. The Hilbert space for the $(\rho, s = \frac{1}{2})$ UIR is the space of Dirac bispinors $\psi(\vec{p})$ which satisfy the Dirac equation. In the Dirac representation of γ-matrices,

$$\psi(\vec{p}) = \begin{pmatrix} \varphi(\vec{p}) \\ -\dfrac{\vec{p}\cdot\vec{\sigma}}{p_0 + m}\,\varphi(\vec{p}) \end{pmatrix} . \tag{19}$$

The scalar product is given by

$$(\psi, \psi') = \int \frac{d^3p}{2p_0}\,\psi^\dagger\gamma^0\psi' = \int \frac{d^3p}{p_0}\,\frac{2m}{p_0 + m}\,\varphi^\dagger\varphi' . \tag{20}$$

Having an explicit representation, we can solve the eigenvalue problems of the coordinates and thus determine properties of the space. Because of the specific scalar product, hermiticity of coordinates is sometimes not completely obvious. Introducing the expressions for the generators, we find

$$W^0 = -\frac{1}{2m}\begin{pmatrix} (\rho - \frac{i}{2})p_i\sigma^i + i\,p_0^2\,\frac{\partial}{\partial p^i}\sigma^i & \epsilon^{ijk}p_0 p_i\,\frac{\partial}{\partial p^j}\,\sigma_k + \frac{3i}{2}p_0 \\ \epsilon^{ijk}p_0 p_i\,\frac{\partial}{\partial p^j}\,\sigma_k + \frac{3i}{2}p_0 & (\rho - \frac{i}{2})p_i\sigma^i + ip_0^2\,\frac{\partial}{\partial p^i}\,\sigma^i \end{pmatrix} \tag{21}$$

$$W^4 = -\frac{1}{2}\begin{pmatrix} ip_0\,\frac{\partial}{\partial p^i}\sigma^i & \epsilon^{ijk}p_i\,\frac{\partial}{\partial p^j}\,\sigma_k + \frac{3i}{2} \\ \epsilon^{ijk}p_i\,\frac{\partial}{\partial p^j}\,\sigma_k + \frac{3i}{2} & ip_0\,\frac{\partial}{\partial p^i}\sigma^i \end{pmatrix} . \tag{22}$$

To find the spectrum of the embedding time W^0 we do not have to solve a differential equation: from the matrix elements of $M_{\alpha\beta}$ one can easily find that the spectrum of W^0 is discrete in all UIR's, with eigenvalues $k(k+1) - k'(k'+1)$.

Because of symmetry, the spectra of spatial coordinates W^i and W^4 are the same: as simpler, we solve the differential equation for W^4. It commutes with spatial rotations so we can take the Ansatz of the form

$$\varphi(\vec{p}) = \frac{f(p)}{p}\,\varphi_{jm} + \frac{h(p)}{p}\,\chi_{jm}\,, \tag{23}$$

where φ_{jm} and χ_{jm} are spinor spherical harmonics,

$$\varphi_{jm} = \begin{pmatrix} \sqrt{\frac{j+m}{2j}}\,Y^{m-1/2}_{j-1/2} \\ \sqrt{\frac{j-m}{2j}}\,Y^{m+1/2}_{j-1/2} \end{pmatrix}, \quad \chi_{jm} = \begin{pmatrix} \sqrt{\frac{j+1-m}{2(j+1)}}\,Y^{m-1/2}_{j+1/2} \\ -\sqrt{\frac{j+1+m}{2(j+1)}}\,Y^{m+1/2}_{j+1/2} \end{pmatrix}, \tag{24}$$

and Y_l^m are spherical harmonics in momentum space, $p = |\vec{p}|$, $j = \frac{1}{2}, \frac{3}{2}, \ldots$.

The W_4-eigenvalue equation for bispinor $\tilde{\psi}_{\sigma jm}$, (19), (denoted by tilde to distinguish it from eigenfunctions of τ in the next section)

$$W_4\,\tilde{\psi}_{\sigma jm} = \sigma\tilde{\psi}_{\sigma jm}\,, \tag{25}$$

reduces to two coupled equations for spinors $\tilde{\varphi}_{\sigma jm}$. Introducing

$$\tilde{f} = (x^2-1)^{1/4}\,\tilde{F}\,, \qquad \tilde{h} = (x^2-1)^{1/4}\,\tilde{H}\,, \tag{26}$$

and variable $x = p_0/m \in (1,\infty)$, we obtain a set of Legendre equations

$$(x^2-1)\,\frac{d^2\tilde{F}}{dx^2} + 2x\,\frac{d\tilde{F}}{dx} - \frac{j^2}{x^2-1}\,\tilde{F} = 2i\sigma(2i\sigma-1)\tilde{F}\,, \tag{27}$$

$$(x^2-1)\,\frac{d^2\tilde{H}}{dx^2} + 2x\,\frac{d\tilde{H}}{dx} - \frac{(j+1)^2}{x^2-1}\,\tilde{H} = 2i\sigma(2i\sigma-1)\tilde{H} \tag{28}$$

with a relation between $\tilde{F}$ and $\tilde{H}$. A regular solution to these equations exists for every real σ, and is expressed in terms of the associated Legendre functions:

$$\tilde{f}_{\sigma j} = A\,(x^2-1)^{\frac{1}{4}}\,P^{-j}_{-2i\sigma}(x), \qquad \tilde{h}_{\sigma j} = A\,(2i\sigma-j-1)(x^2-1)^{\frac{1}{4}}\,P^{-j-1}_{-2i\sigma}(x) \tag{29}$$

Eigenfunctions of W^4 are orthogonal and normalized to δ-function,

$$\big(\tilde{\psi}_{\sigma jm}, \tilde{\psi}_{\sigma' j' m'}\big) \;=\; 2A^*A'\frac{\Gamma(\frac{1}{2}-2i\sigma)\,\Gamma(\frac{1}{2}+2i\sigma')}{\Gamma(j+1-2i\sigma)\,\Gamma(j+1+2i\sigma')}\delta_{mm'}\,\delta_{jj'}\,\delta(\sigma-\sigma')\,. \tag{30}$$

Thus, the spectrum of W^4 (and all W^i) is continuous, the real line.

5. Cosmic time

We can use the same Ansatz to analyze the eigenvalues of the cosmic time τ, $e^{\tau/\ell} = W_0 - W_4$, i.e. to solve

$$(W_0 - W_4)\,\psi_{\lambda jm} = \lambda\psi_{\lambda jm}\,. \tag{31}$$

After some calculation, we find that now the natural variable in which differential equations simplify is $z = \sqrt{\frac{p_0-m}{p_0+m}} \in (0,1)$. We introduce

$$f = \left(\frac{2}{1-z^2}\right)^{-i\rho} z^{j+\frac{1}{2}}\,F\,, \quad h = \left(\frac{2}{1-z^2}\right)^{-i\rho} z^{-j-\frac{1}{2}}\,H \tag{32}$$

and obtain the following of Bessel equations

$$z^2\,\frac{d^2F}{dz^2} + z\,\frac{dF}{dz} + \left(4\lambda^2 z^2 - j^2\right)F = 0$$
$$z^2\,\frac{d^2H}{dz^2} + z\,\frac{dH}{dz} + \left(4\lambda^2 z^2 - (j+1)^2\right)H = 0\,.$$

The regular solution to these equations is given by

$$f_{\lambda j} = C\left(\frac{2}{1-z^2}\right)^{-i\rho}\sqrt{z}\,J_j(2\lambda z), \quad h_{\lambda j} = iC\left(\frac{2}{1-z^2}\right)^{-i\rho}\sqrt{z}\,J_{j+1}(2\lambda z). \tag{33}$$

It exists for every real λ, but for λ and $-\lambda$ the functions are proportional. So apparently, $\lambda \in (0,\infty)$, and the spectrum of $W_0 - W_4$ is continuous. However, when we calculate the scalar product

$$(\psi_{\lambda jm}, \psi_{\lambda' j' m'}) \sim \delta_{jj'}\delta_{mm'}\int_0^1 z dz\,\left(J_j(2\lambda z)J_j(2\lambda' z) + J_{j+1}(2\lambda z)J_{j+1}(2\lambda' z)\right) \tag{34}$$

we find that it is bounded for $\lambda = \lambda'$, i.e. all solutions are normalizable, which is in contradiction with the statement that they belong to a continuous spectrum. The solutions are also not orthogonal for $\lambda \neq \lambda'$.

Thus, not all of formal solutions (33) can be eigenfunctions, that is, $W_0 - W_4$ is not a self-adjoint operator. One can perform a detailed analysis of hermiticity [3], and find that the domain $\mathcal{D}(W_0 - W_4)$ is unequal to the domain of the adjoint operator, $\mathcal{D}(W_0^\dagger - W_4^\dagger)$. The corresponding deficiency indices are equal, $(n_+, n_-) = (1,1)$. That means that it is possible to find self-adjoint extensions by restricting the domain of $W_0^\dagger - W_4^\dagger$, [12].

The appropriate condition which defines the domain is

$$F(0) = H(0) = 0, \qquad H(1) = icF(1)\,, \tag{35}$$

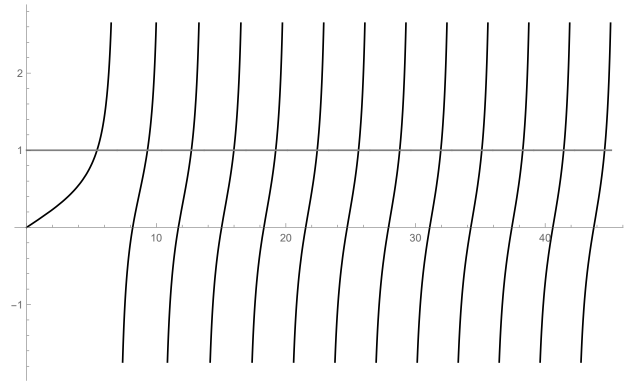

Figure 1: Solutions to Eq. (36) for $j = \frac{7}{2},\ c = 1$.

One can see that eigenfunctions (33) can satisfy (35). First relation is clearly true, the second gives

$$\frac{J_{j+1}(2\lambda)}{J_j(2\lambda)} = c = \text{const}, \tag{36}$$

that is, an equation for λ. This equation, as seen from Figure 1, has infinitely many solutions for every real c; the set of solutions is discrete. Using the recurrence relations between the Bessel functions, it is possible to show that these solutions orthonormal. We thus conclude thae cosmic time has discrete spectrum.

6. Initial singularity

There are several immediate consequences of the above considerations. We can verify that fuzzy de Sitter space corresponds to an expanding cosmology. The (squared) radius of the universe is given by

$$(x^i)^2 = -\ell^2 W_i W^i \tag{37}$$

and its evolution can be traced by the expectation value $\langle (x^i)^2 \rangle$ in the eigenstates of time. Using the Casimir relation

$$-W_i W^i = \mathcal{W} + W_0^2 - W_4^2\,, \tag{38}$$

and taking the expectation value in normalized eigenstates $\psi_{\lambda jm}$, we find

$$\langle -W_i W^i \rangle = \mathcal{W} + \langle (W_0 + W_4)(W_0 - W_4) \rangle = \mathcal{W} + \lambda^2 + 2\lambda \langle W_4 \rangle\,.$$

The value of $\langle W_4\rangle$ can be estimated: analyzing the explicit expressions we obtain $0 \le \langle W_4\rangle \le \frac{\lambda}{2}$, and hence

$$\mathcal{W} + \lambda^2 \le \langle -W_i W^i \rangle \le \mathcal{W} + 2\lambda^2 . \tag{39}$$

The expectation value of the radius of the universe is bounded below by $\ell\sqrt{\mathcal{W}}$: it cannot vanish in physical states. This fact resolves the problem of the initial big bang singularity. The radius grows exponentially with time: for late times we have $\sqrt{\langle -W_i W^i\rangle} \sim \lambda = e^{\langle\tau\rangle/\ell}$.

Discreteness obtained by requiring self-adjointness is known in other cases of quantum spaces too, [13, 14, 15, 16]. It becomes relevant in the 'deep quantum region' $\lambda \to 0$, i.e. $\langle\tau\rangle \to -\infty$, i.e. near the big bang. For values away from the Planck scale time is almost continuous: the difference between its consecutive eigenvalues is macroscopically negligible,

$$\tau_{n+1} - \tau_n \approx \ell \, \log\left(1 + \frac{1}{n}\right) . \tag{40}$$

Discretization of time means also that the initial classical symmetries of fuzzy de Sitter space near the Planck scale are spontaneously broken, by the choice of the self-adjoint extension. As we can see, it is restored at large scales. It will be important to understand, in the future, the nature of the symmetry breaking, and whether perhaps symmetry remains as deformed or quantum symmetry.

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