Fermions, hairy blackholes and hairy wormholes in anti-de Sitter spaces*

Mihailo Čubrović[†]
Center for the Study of Complex Systems,
Institute of Physics Belgrade, Serbia

Abstract

We discuss the existence, properties and construction (analytical and numerical) of hairy black holes with fermionic matter in asymptotically anti-de-Sitter space. The negative cosmological constant makes hairy black holes stable, and the nucleation mechanism can make the formation of hair at the horizon energetically and entropically preferable to conventional black holes. The difficulties intrinsic to fermions at finite density – the Pauli principle and exchange interactions – require some drastic approximations in calculating the stress-energy tensor and geometry. We will consider several methods on the market – Hartree-Fock, WKB, and fluid-mechanical methods, and consider the dual field theories of these constructions. Then we will apply the same methods to the construction of wormholes; fermions are a natural candidate for wormhole source matter as they have a Dirac sea of negative energies, and negative energy-momentum density is the condition for wormhole formation. The field theory interpretation of wormholes is still open but has to do with strongly entangled systems. The paper combines a pedagogical introduction to the basic methods and results (obtained in the last 10+ years) with an account of fresh research results, mainly on the wormhole applications and non-planar black holes.

1. Introduction

AdS black holes are a favorite topic, not only in relation to holography but also in general: AdS space behaves like a potential box, the cosmological constant provides an effective repulsive force at large distances and the existence of a boundary at spatial infinity makes bound states possible. All of

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† e-mail address: mcubrovic@gmail.com

this brings about the famous result that hairy black holes are indisputably possible, and well-studied. In full (global) AdS space, one may have small black holes, which barely see the boundary and radiate like in asymptotically flat space, and large black holes, which reach an equilibrium state with the Hawking radiation at given temperature and remain stable forever (eternal AdS black holes). We will focus on the latter, as they can be treated as (semi)classical stationary systems. Clearly, just like the Hawking radiation, matter and gauge fields can likewise equilibrate between the black hole horizon and AdS boundary, possibly forming hair – by definition, it means nonzero density of some field (and possibly nonzero expectation values of other operators, like charge density, spin, etc) at the horizon itself. This in turn means that the geometry changes as opposed to the no-hair case: the hair itself enters the stress-energy tensor, and the outcome is a hairy black hole geometry, where a horizon still exists but with a different metric. At zero temperature, hair tends to remove extremal black holes in favor of zero-area horizons, with zero Bekenstein-Hawking entropy. We will soon discuss several explicit examples of this phenomenon.

The above story acquires an additional dimension thanks primarily to the AdS/CFT correspondence (gauge/gravity duality) [1, 2, 3] – the fact that the bulk gravity physics is equivalent to a quantum field theory in flat space in one dimension less, whose operators act as boundary sources of the AdS (bulk) fields. The actions in AdS (with field Φ) and in CFT (with field O, which acts as a boundary source to Φ) are equal:

$$\begin{aligned} S_{\text{AdS}} &= S_{\text{CFT}} \\ S_{\text{AdS}} &= \int \mathcal{D}\Phi \exp\left(-\int_{\text{AdS}} d^{D+1}x\sqrt{-g}\mathcal{L}_{\text{AdS}}\left(\Phi, \partial_\mu\Phi\right) + \oint_\partial d^D x\sqrt{-h}\mathcal{O}\Phi\right) \\ S_{\text{CFT}} &= \int \mathcal{D}\mathcal{O} \exp\left(-\int d^D x\mathcal{L}_{\text{CFT}}\left(\mathcal{O}\right)\right), \end{aligned} \tag{1}$$

where we have denoted by ∂ the boundary of the AdS space, $g_{\mu\nu}$ is the AdS metric and $h_{\mu\nu}$ is the induced metric at the boundary. From now on, integrals over the bulk of AdS will be dnoted just by $\int$, understanding that the integral is over the whole space. At this place we do not intend to explain AdS/CFT and its applications in any detail; suffice to say that one can obtain thermodynamic potentials and correlation functions in field theory, which has found important applications in condensed matter theory, quantum chromodynamics and conformal field theory. Interested readers can consult [4, 5, 6] for reviews. In this work we deal with the gravity side of the correspondence – the formation of a hairy black hole with fermionic matter, which corresponds to a finite electron density phase in field theory. We assume the familiarity with the basic notions of AdS space and quantum field theory in curved spacetime, for example at the level of [7] and [8], respectively.

Mathematically, the topic of this review is the solution of the coupled Einstein-Maxwell-Dirac system with the total action $S_{\text{AdS}} = S_{\text{bulk}} + S_\partial$.

The bulk action reads:

$$
\begin{aligned}
S_{\text{bulk}} &= S_{\text{E}} + S_{\text{M}} + S_{\text{Dir}} \\
S_{\text{E}} &= \int d^4x\sqrt{-g}\,(R+6) \\
S_{\text{M}} &= -\int d^4x\sqrt{-g}\frac{\hat{F}^2}{4} \\
S_{\text{Dir}} &= -\int d^4x\sqrt{-g}\left(\frac{1}{2}\bar{\Psi}D_\mu e^\mu_a\Gamma^a\Psi + \frac{1}{2}\bar{\Psi}e^\mu_a\Gamma^a\Psi + m\bar{\Psi}\Psi\right).
\end{aligned}
\tag{2}
$$

Here, $\hat{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic (EM) field strength tensor, and the the cosmological constant in AdS_4 is $6/L^2$, where the AdS radius $L = 1$ is set to unity, as we will mainly work on the Poincare patch of AdS space, so all other dimensionful quantities can be expressed in terms of L. The Dirac bispinor Ψ has mass m and charge q, and the covariant derivative

$$
D_\mu = e^a_\mu D_a = \partial_\mu - \frac{\imath}{8}\left[\Gamma^a, \Gamma^b\right]\omega^\mu_{ab} - \imath q A_\mu \tag{3}
$$

depends on the spin connection ω^μ_{ab} and the gauge field A_μ, and the gamma matrices satisfy the usual relations $\left[\Gamma^a, \Gamma^b\right] = 2\eta^{ab}$, with the Minkowski metric η. We will be using the mostly plus convention. Obviously, $\Psi = 0$ is a solution, and in this case we get a Schwarzschild black hole if the EM field is also zero, or a charged Reissner-Nordstrom (RN) black hole for nonzero field strength. The question is, are there other solutions, with nonzero profile Ψ? Such solutions describe hairy black holes at finite temperature: the horizon is typically still there, but the geometry is changed. At zero temperature, the black hole might disappear. Since AdS space has a boundary, there is also a boundary contribution to the action, as in (1), depending on extrinsic curvature K, boundary cosmological constant λ and the boundary values of the fields:

$$
S_\partial = \oint_\partial d^3x\sqrt{-h}\left[K - \lambda - \frac{1}{2}n_\mu A_\nu \hat{F}^{\mu\nu} - \frac{1}{2}\bar{\Psi}\Psi\right]. \tag{4}
$$

The classical equations of motion do not depend on the boundary action. However, S_∂ is still important (1) to make sure there is a good action principle, i.e., that the on-shell solutions are indeed minima of the action[1] (2) to regularize any UV divergences (3) to get correct thermodynamics. The last point will be particularly important: one way to see that the hairy black hole and not the bald black hole is the true vacuum will be the fact that the action on the hairy solution is lower.

Solving the system (2) is a problem in quantum field theory at finite density. We work with classical general relativity (GR) and classical EM

[1] Remember that the (bulk) Euler-Lagrange equations are only a necessary condition for the minimum of the action.

field, but *fermions are never classical*; this is the first important lesson. The Pauli principle always introduces nonlocal correlations which show as the exchange interaction. Another way of saying this is that the pressure of a fermionic gas or fluid always includes the quantum contribution which is absent in both classical and bosonic gas; that is the reason that organized matter such as stars, planets, chairs and notebooks has rigidity and does not collapse onto itself. Therefore, even though we do gravity at $\hbar = 0$, the fermions even at leading order need to be tackled quantum-mechanically. This means calculating the *fermionic determinant*:

$$Z_{\mathrm{Dir}} = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi e^{-S_{\mathrm{Dir}}} \text{ ``} = \text{''} \ \left[\det\left(D_\mu e^\mu_a \Gamma^a + e^\mu_a \Gamma^a D_\mu + m\right)\right]^{1/2}. \tag{5}$$

We have put the equality sign under quotation marks because the determinant is actually the product of the eigenenergies of all the modes (an infinity of them), which is not only badly divergent (that could be regulated) but is also impossible to calculate because of the *fermion sign problem*, the fact that the fermionic modes enter the path integral with a sign that can be plus or minus. This makes the measure in the path integral (5) non-probabilistic and makes it impossible to expand around a classical solution in a controlled way. Fortunately, the AdS metric turns out to simplify the problem enough that it can be tackled in a way which is tractable and, while of course not exact, can be systematically improved in a perturbative way. This is in fact the motivation behind AdS/CFT modelling of strongly correlated electron systems: the fermion sign problem is fatal for strongly coupled field theories in flat space, but in GR with AdS boundary conditions it transforms into a difficult but doable task.

Is the journey worthwhile? In line with the broad scope of the Belgrade Mathematical Physics Meetings, we have anticipated a broad readership of this paper and thus we have decided to give a very general and perhaps rather dry introduction to the topic of fermionic hairy black holes. This necessarily means that we will not touch upon the many interesting applications: AdS/CFT and its applications to quantum chromodynamics and condensed matter physics, the black hole information problem, the critical phenomena in gravitational collapse and the black hole solutions in string theory. We do discuss one special topic that we currently find very interesting: hairy wormholes generated by fermion matter, where many of the methods used for hairy black holes can be successfully applied. The main task of the paper is to provide a tutorial on the basic methodology and calculation techniques, bringing the reader to the point that he can understand and repeat the calculations from the literature and start doing his own. The existing literature is rather heterogenuous and there is no single text to recommend. We will give the references we deem particularly useful throughout the paper, without the pretention of being exhaustive; the choice of references is certainly dictated also by our prejudices and tastes.

Plan of the paper. In Section 2 we first explain the instabilities of AdS space and AdS black holes to a nonzero density profile of fermions, and introduce the basic concepts that will keep appearing throughout the paper: effective potential and the bound states of the fermionic wavefunctions. In Section 3 we first treat the problem in the consistent one-loop (Hartree-Fock) approximation, calculating the determinant (5) by definition, from the individual wavefunctions for different states. We find this job surprisingly difficult – it is still an active research area. But we are able to give a qualitative picture of the outcome and sketch the phase diagram, depending on the chemical potential μ and fermion mass and charge m, q. As we move toward the high-fermion-density corner of the phase diagram, the things simplify. The simplest and "most classical" limit of the problem is the limit of large density. It is a rule of thumb that for fermions, the role of interactions diminishes as the density grows. At high density, WKB approximation works very well. At highest densities, we find semiclassical fluid with an equation of state that takes into account the fermionic pressure, similar in spirit to the Oppenheimer-Volkov equations for neutron stars. In section 4 we apply these methods to a different topic – hairy wormholes instead of black holes. This problem has recently gained notoriety and might carry some important messages for the black hole information problem. The final section sums up the conclusions.

2. Planar AdS black holes and fermion nucleation

In this and the next section we will focus on large planar black holes on the Poincare patch of AdS space. Large black holes can reach equilibrium with the AdS boundary so they do not emit Hawking radiation and can exist eternally. The Poincare patch of AdS_4 space is a coordinate chart with a single boundary on one end and interior on the other end. It does not cover the whole AdS space but is simpler to work with than global AdS and is good enough to desribe the instability at the horizon. The metric of pure AdS space witout a black hole is given by

$$ds^2 = r^2\left(-dt^2 + d\vec{x}^2\right) + \frac{dr^2}{r^2} = \frac{1}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2\right) \tag{6}$$

where $r = 1/z$ is the radial coordinate, t is time and $\vec{x} = (x, y)$ are the transverse coordinates. The AdS boundary is at $r = \infty$ ($z = 0$), and the interior is at $r = 0$ ($z = \infty$). From now on we will mainly use the z coordinate; we will always specify explicitly if a different radial coordinate is used. In AdS/CFT, the radial coordinate corresponds to the energy scale in field theory: the near-boundary region encodes for the physics at high energies, in the ultraviolet (UV), and the deep interior, with z large, is the infrared (IR). Even though we do not consider the CFT dual here, we will still adopt the UV/IR terminology.

In the presence of a point electric charge e we get a Reissner-Nordstrom (RN) black hole with the horizon at $z_h = 1$, with charge e, mass M and temperature T:

$$ds^2 = \frac{1}{z^2}\left(-f(z)dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)}\right), \quad f(z) = 1 - Mz^3 + e^2 z^4$$

$$M = z_h^3 + e^2, \quad A = \frac{ez_h}{2\sqrt{\pi}}(1 - z/z_h)dt, \quad T = \frac{3z_h}{4\pi}\left(1 - \frac{e^2}{3z_h^4}\right) \tag{7}$$

For $e = 0$ we get the Schwarzschild AdS black hole, and for $e = \sqrt{3}z_h^2$ the black hole becomes extremal, with temperature $T = 0$. To see this, remember that the black hole temperature is given by $f(z \to z_H) = 4\pi T(z - z_H) + \ldots$, so plugging in f from above we indeed get the correct expression for T. Importantly, the near-horizon region of a black hole is an AdS space [7]. This IR AdS space (near $z = z_h$) has a priori nothing to do with the AdS asymptotics in the UV (near $z = 0$); it is there also for black holes in flat or dS space. At $T = 0$, rescaling $z - e/\sqrt{3} \mapsto 1/6\epsilon\xi$ and expanding in ϵ to lowest order gives the metric

$$ds^2 = \frac{1}{6}(-dt^2 + d\xi^2) + \frac{e^2}{3}d\vec{x}^2, \quad A_t = \frac{1}{\sqrt{6}\xi}. \tag{8}$$

The is $\mathrm{AdS}_2 \times R^2$ geometry, a direct product of AdS with a plane. At finite temperature, a similar rescaling can be worked out, yielding again an AdS_2 throat. Since the throat describes the near-horizon region, instabilities of the black hole can be figured out from possible instabilities of this IR AdS space. Once again, this is *not* the whole AdS_4, which is always stable far from the horizon, in the UV (otherwise our whole classical gravity approach crumbles down), it is just a region near the horizon, in IR.

In order to write the equations of motion, we have to choose a basis for the gamma matrices and the form of the Dirac bispinor (remember that only two out of four components are really independent degrees of freedom). A convenient representation is

$$\Gamma^0 = \sigma^1 \otimes \imath\sigma_2, \ \Gamma^1 = \sigma^1 \otimes \sigma_1, \Gamma^2 = \sigma^1 \otimes \sigma^3, \ \Gamma^z = \sigma^3 \otimes \hat{1}. \tag{9}$$

so that the Dirac equation in a spherically symmetric metric defined as $\mathrm{diag}(g_{tt}, g_{ii}, g_{ii}, g_{zz})$ gives two equivalent decoupled pairs of equations. Taking the Dirac bispinor in the form $\Psi = (\psi_1, \chi_1, \imath\chi_2, \imath\psi_2)^T$, the equations for $\psi_{1,2}$ read [9, 10]:[2]

$$\partial_z\psi_{1,2} \pm \hat{m}\psi_{1,2} - \left(\mp\hat{E} + \hat{k}\right)\psi_{2,1} = 0 \tag{10}$$

$$\hat{m} \equiv m\sqrt{g_{zz}}, \hat{\mu} \equiv \sqrt{\frac{g_{zz}}{-g_{tt}}}A_t, \ \hat{E} \equiv q\hat{\mu} + E\sqrt{\frac{g_{zz}}{-g_{tt}}}, \ \hat{k} \equiv \sqrt{\frac{g_{zz}}{g_{ii}}}k. \tag{11}$$

[2]Since only two components of the Dirac bispinor are independent, the system for $\chi_{1,2}$ yields no new information.

We have Fourier-transformed the derivatives over time and transverse spatial dimensions as $\partial_t = -\imath\omega, \partial_x = \imath k_x, \partial_y = \imath k_y$, and we have exploited the spherical symmetry to set $k_x = k, k_y = 0$. The quantities $\hat{E}$, $\hat{k}$, $\hat{\mu}$ can be informally interpreted as "local" values of the energy, momentum and chemical potential, respectively. The "local" values equal E, k, μ at the AdS boundary, grow monotonously toward the horizon and diverge there, a consequence of the infinite redshift seen by a faraway observer. An important idea is to consider the Schrödinger form of the Dirac equation instead, differentiating (10) once with respect to z, decoupling the equations for $\psi_{1,2}$, and elliminating the first derivatives $\psi'_{1,2}$ by introducing the tortoise coordinate s instead of z. The resulting picture is that of a zero-energy Schrödinger equation, of the form $\partial_s^2\psi_{1,2} - V_{\mathrm{eff}}(s)\psi_{1,2} = 0$, in an effective potential $V_{\mathrm{eff}}(s)$.[3] Near the horizon, the potential is constant at leading order [11]:

$$V_{\mathrm{eff}}(s \to -\infty) = \frac{m^2 + 12k^2/\mu^2 - 2q^2}{(q/\sqrt{2} + k)^2} + \dots \tag{12}$$

It is true that the Schrödinger form is only a consequence of the Dirac equation, not equivalent to it: extra conditions must be imposed on the Schrödinger solution to make it satisfy the Dirac equation. But the effective potential is great for qualitative insights and it contains the basic idea of the black hole instability in a very transparent way. The near-horizon potential can contain bound states if it is negative, hence the instability criterion for a fermionic mode with momentum k is that the numerator of (12) is negative. Fermions fill up the potential well starting from $k = 0$ up to some maximum k for which (12) reaches zero. Therefore, the instability first sets in when V_{eff} is negative for $k \to 0$, so we get our first rule-of-thumb prediction: the black hole will be surrounded by a gas of fermions and become hairy when

$$m < q\sqrt{2}. \tag{13}$$

But this is just one end of the potential well; what happens at the other end? Plugging in the pure AdS metric (6) into (12) we get

$$V_{\mathrm{AdS}}(s \to 0) = \frac{m^2 + m + k^2}{(k + \mu)^2}\frac{1}{s^2} + \dots, \tag{14}$$

which is always non-negative, and grows to infinity. This is good – there is never an instability in the far UV, and the fermionic hair can never come arbitrarily close to the AdS boundary. It also means that bound states in the interior will indeed exist whenever (13) is negative. The physical picture is the following: in the presence of EM and gravitational field of the black hole, fermions are pair-created. These pairs are virtual, and

[3] This is a simple exercise that we will do many times; the reader should be able to do the necessary (straightforward) calculations leading to the expression for $s(z)$ and $V_{\mathrm{eff}}(s)$.

they only have a finite probability of becoming long-living if the external potential energy is large enough. In that case, bound states form, and there is a solution of (2) with nonzero fermion density. In the literature, this is sometimes called fermion nucleation. For scalars, similar logic leads to the Breitenlohner-Freedman bound, which puts a constraint on the scalar mass for the stability of the UV (with fermions, as we have seen, UV is always stable), and in IR it similarly gives a criterion for forming hair [5]. We also see from (14) and Fig. 1 that the potential well becomes shallower as k grows, so the bound states only exist up to some maximum $k = k_F$ which is really the Fermi momentum of the bulk Fermi sea.

From (12,14) we can understand the behavior of the effective potential. In Fig. 1, we give the function $V(s)$ in the whole space, from $z = 0$ ($s = 0$), to $z = z_h$ ($s = -\infty$). The fermionic modes fill the potential well until they reach the energy $E = 0$. From (12), higher modes correspond to higher momentum k. The fermionic density is thus given by a sum over these bound states. The easiest case is in fact an extremely deep well: the energy levels are so dense and so numerous that they can be approximated by a continuum; this is called electron star limit. But the most interesting regime is the one with only a few wavefunctions, which really describes the transition to a hairy solution. This is a much harder nut to crack.

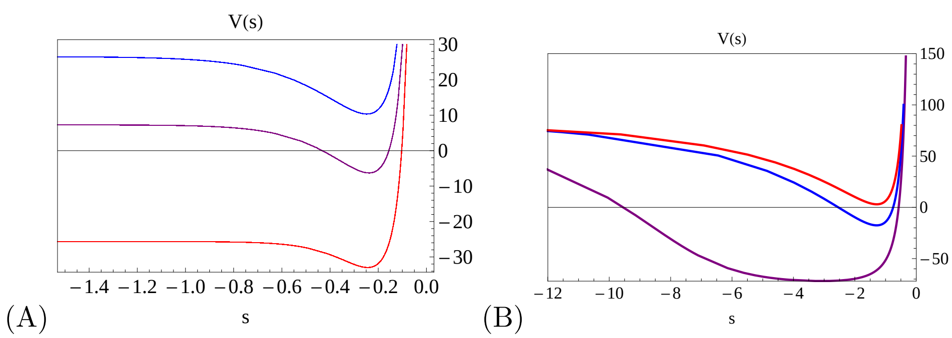


Figure 1: Effective potential V_{eff}, as a function of the tortoise coordinate $s \equiv \int_0^z dz g_{zz}(z)$, in the RN metric (A), and in the Lifshitz metric (B), for $q = 1$, $m = 0.4$, $\mu = 1$, and three momentum values increasing from violet to blue to red: $k = 1, 2, 3$ (A) and $k = 0, 5, 10$ (B). In both cases, the negative potential well becomes shallower and shallower and eventually disappears as k grows, so we fill the bulk Fermi sea up from $k = 0$ to some maximal $k = k_F$. In the black hole background, the potential is flat for $s \to -\infty$, which corresponds to the AdS_2 near-horizon region and signifies an instability as the bound states extend all the way to the horizon ($s = -\infty$). In the backreacted Lifshitz metric the potential grows for $s \to -\infty$, suggesting that deep IR is stable: the true vacuum is the Lifshitz geometry, not RN. Taken over from [14].

3. Fermionic hair

Now that we have convinced ourselves that hairy solutions, with finite fermion density, have to exist, we need to solve the full system of Einstein, Maxwell and Dirac equations to find them. Clearly, a more general ansatz for the metric than (7) is needed now, and we will write it as

$$ds^2 = -\frac{F(z)G(z)}{z^2}dt^2 + \frac{1}{z^2}d\vec{x}^2 + \frac{1}{F(z)z^2}dz^2, \tag{15}$$

leading to Einstein-Maxwell equations

$$1 - F + zF'/3 - T^{\rm tot}_{tt}FG/3z^2 = 0 \tag{16}$$

$$G' + z(T^{\rm tot}_{tt}/F^2 + T^{\rm tot}_{zz}G) = 0 \tag{17}$$

$$A_t'' - G'/2GA_t' + qn\sqrt{G}/\sqrt{F}z^3 = 0, \tag{18}$$

where $T^{\rm tot}_{\mu\nu}$ is the total stress-energy tensor, both from the electric field (which is easy to find) and from the fermions (which is our big problem). A typical situation in hairy problems is that formulating the physically meaningful boundary conditions is not so easy. Notice the Einstein equations are first-order, so we need one boundary condition for each function (F and G), whereas the Maxwell equation is second-order and requires two boundary conditions. Let us now summarize what boundary behavior we expect on physical grounds.

1. The AdS asymptotics for the metric and gauge field require $F(z \to 0), G(z \to 0) = 1, A_t(z \to 0) = \mu$. So far it's all simple.
2. The main puzzle for the IR geometry is – does the horizon disappear or not? At $T = 0$ we do not expect that the degenerate RN horizon can survive. So we do not expect zeros in F, G but we do expect their derivatives to vanish in order to have a smooth solution (finite derivatives at $z \to \infty$ would likely give divergent curvature). Thus at $T = 0$ we need $F'(z \to \infty) = G'(z \to \infty) = 0$ or, in other words, $F(z \to \infty) = \text{const.} + O(1/z)$ and likewise for G. At finite temperature, general GR arguments suggest there is a horizon at some $z = z_h$ satisfying $F'(z \to z_h) = 4\pi T$.
3. The IR behavior of the gauge field is related to the question: *is all the charge carried by the fermions, or the charge is shared between the fermions and the horizon*? The Gauss-Ostrogradsky theorem for the AdS space, with a UV boundary and either a horizon or a smooth far-away IR takes the form [12]:

$$\oint_\partial d^3x\sqrt{-h}|_{z\to 0} \star \hat{F} = \int d^4x\sqrt{-g}qn + \oint_{\rm IR} d^3x\sqrt{-h_{\rm IR}}|_{z=z_{\rm IR}} \star \hat{F} \tag{19}$$

Here, $\star\hat{F}$ is the coordinate-invariant flux of the 2-form $\hat{F}$, and $h_{\rm IR}$ is the induced metric on the surface normal to the radial direction at

$z_{\rm IR} = z_h$ or $z_{\rm IR} = \infty$, depending on whether there is a horizon or not. In principle, the IR charge might be shared between the horizon and the fermions. However, we will find that in the semiclassical calculation there are no solutions where the charge is shared – any backreaction will always expell all the charge from the IR.

4. The boundary conditions for the Dirac equation present no problems and are pretty standard in AdS space [13]. In the UV, out of the two branches, we want the subleading one, with the motivation to preserve the AdS asymptotics, i.e., to perturb the space as little as possible in the UV. In particular, the near-boundary expansion of (11) gives

$$\psi_1(z \to 0) = \frac{E + \mu q - k}{2m - 1} A_2 z^{5/2-m} + B_1 z^{3/2+m} + \dots$$
$$\psi_2(z \to 0) = A_2 z^{3/2-m} + \frac{E + \mu q + k}{2m + 1} B_1 z^{5/2+m} + \dots, \quad (20)$$

so we pick $A_2 = 0$, as the leading contribution for $z \to 0$ comes from the $z^{3/2}$ term. In the IR, the metric determines the boundary conditions: if there is a horizon, we need $\Psi(z = z_h) \to 0$ for stability, if not, then to avoid infinite energy density at large z we require $\partial_z \Psi(z \to \infty) = 0$, for otherwise a nonconstant density profile would give rise to a diverging curvature. The attentive reader should be alarmed: this means two boundary conditions for each component (one in UV and one in IR), but the equations are only first-order. The resolution is that for given momenta, the energy is not arbitrary but fixed by the dispersion relation $E(k)$; thus solving the Dirac equation in an effective potential well introducs energy quantization, as one would expect.

What remains is to find the fermionic stress tensor. Since spinors couple to the spin connection e^μ_a and not directly to the metric, the stress tensor is expressed as

$$T_{\mu\nu} = \left\langle \frac{1}{4} e_{\mu a} \bar{\Psi} \Gamma^a D_\nu \Psi + (\mu \leftrightarrow \nu) \right\rangle, \quad (21)$$

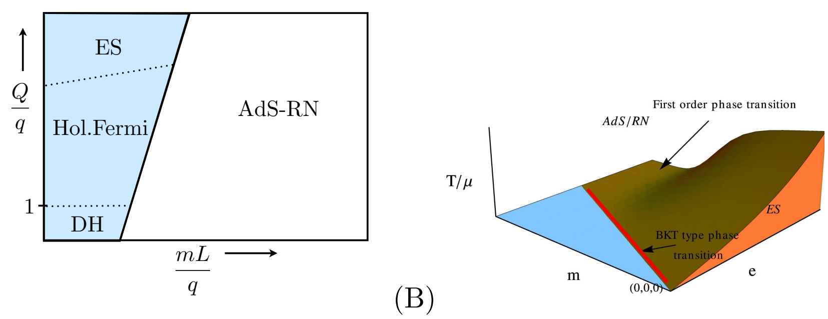
and the expectation value $\langle \dots \rangle$ reminds us that the fermions are never classical. At zero temperature, the state is pure and can be represented as the sum of (appropriately normalized) radial modes with energies E_ℓ, where ℓ is the radial quantum number, and the energies E_ℓ are all ≤ 0. At finite temperature, the state is mixed and gets a contribution from both positive and negative energies E_ℓ, with thermal weights $w_\ell = \exp(-\beta E_\ell)/Z$, the partition sum being $Z = \sum_\ell \exp(-\beta E_\ell)$. With this in mind, we can wrote out (21) as

$$T_{tt} = e_{t0} \sum_{\ell=1}^{N} w_\ell \int_0^{k_F} \frac{k dk}{(2\pi)^2} \left(\psi^\dagger_{1;\ell} \psi_{1;\ell} + \psi^\dagger_{2;\ell} \psi_{2;\ell} \right) (E_\ell + q A_t)$$

$$
\begin{aligned}
T_{ii} &= e_{i1}\sum_{\ell=1}^{N} w_\ell \int_0^{k_F} \frac{kdk}{(2\pi)^2}\left(\psi^\dagger_{1;\ell}\psi_{1;\ell}-\psi^\dagger_{2;\ell}\psi_{2;\ell}\right)k \\
T_{zz} &= e_{z3}\sum_{\ell=1}^{N} w_\ell \int_0^{k_F} \frac{kdk}{(2\pi)^2}\left(\psi^\dagger_{1;\ell}\partial_z\psi_{2;\ell}-\psi^\dagger_{2;\ell}\partial_z\psi_{1;\ell}\right). \qquad (22)
\end{aligned}
$$

For brevity, we write $\psi_{1,2;\ell}\equiv\psi_{1,2}(E\ell,k;z)$. We will consider in detail just the $T=0$ case, when the weights w_ℓ effectively just pick the ground state and cut off all the others, but we will later discuss the results (without details of the calculations) also at finite T. The spectrum is discrete and gapped in the radial direction, so the integral $\int dE/2\pi$ becomes a sum, however in the transverse directions the system remains gapless, filling the whole (spherical) Fermi sea in the $k-$momentum space, as long as the dispersion relation $E(k)=E_\ell\leq 0$ is satisfied for some ℓ. The highest such k, for which $E_\ell=0$, is the Fermi momentum k_F, and the possible momenta are $0\leq k\leq k_F$. It is this continuous quantum number k that makes our life difficult. Here, indeed, our easy path comes to an end, because a self-consistent calculation of the wavefunctions certainly cannot be done in a closed form. Here we must resort to approximations. The number of occupied levels N is a good guide on the kind of approximation one needs to make. One can rephrase it as the ratio Q/q, where Q is the total fermion charge $\int d^4x\sqrt{-g}q\Psi^\dagger\Psi$. The thermodynamic limit, where the number of particles goes to infinity and the charge of an individual fermion to zero so that $N\to\infty, q\to 0, Q=Nq=\text{const.}$, is at one extreme. We expect that the problem approaches the classical regime in this case, and it will turn out to be true. The opposite limit is $Q/q=1$, with just a single excitation, the hairy black hole at birth. We expect this to be likewise a simple limit, however it will turn out not to be quite true. In-between we dial between the quantum mechanics of $N=1$ and the classical field theory of $N\to\infty$ [14].

Phase diagram. Before doing that, we can sum up our qualitative knowledge on a phase diagram (Fig. 2). From (12-14), bound states form for small enough m values (panel (A)); if (13) is valid beyond the probe approximation, the borderline is $m=q\sqrt{2}$. Left of this line there is a hairy solution, to the right of it the AdS_2 near-horizon region (and the whole RN black hole) remain. The hairy solutions are best described in different ways depending on the number of filled levels ($N=Q/q$); this is the topic od the rest of this section. One can also plot the situation at finite temperature (panel (B)). The phases remain the same; more precisely, the extremal black hole becomes a finite-temperature black hole, and the hairy solutions also smoothly develop a hairy horizon (thermal horizon with nonzero fermion density $n(z_h)$). What changes is the order of the phase transition: at $T=0$ it is continuous, and at finite temperature it is discontinuous.



(A) (B)

Figure 2: (A) Phase diagram as a function of the total-to-fermion-charge ratio Q/q (y-axis), and the fermion mass (in units of AdS radius L) over charge ratio mL/q (x-axis). For large masses, the effective potential is positive and the ground state is the bald RN black hole, with quantum critical dual field theory. For smaller masses, hair develops, which corresponds to a Fermi liquid in dual field theory. For $Q \sim q$ (few wavefunctions), the single-wavefunction Dirac hair approximation works; for $Q/q \to \infty$ we approach the semiclassical fluid (electron star) limit; between them there is a smooth crossover with unclear properties, both in AdS and in the holographic dual. Notice different notational conventions for the total charge from the main text (e vs. Q). Taken over from [15]. (B) Adding nonzero temperature as the third axis, we obtain also the thermal phase transitions between the black hole and the hairy solution, which are generically first order, smoothing out to an infinite order (BKT) transition at $T = 0$ – the red line in (B) is the bold black line between the RN and hairy (blue) region in (A).

3.1. Quantum hairy black holes

A controlled approximation is to solve the problem perturbatively, at one-loop order in fermionic fields. This is nothing but the textbook Hartree-Fock (HF) method, but in curved space. Dynamical spacetime makes a big difference: it introduces an additional strongly nonlinear component of the system, making the solution landscape larger and less predictable, and the UV and IR divergences can appear also in the Einstein equations and need explicit regulators. In fact, this is still an open problem – nobody has yet classified the solutions of the Einstein-Maxwell-Dirac system even in the Hartree-Fock approximation, and we do not know what surprises might lurk in this corner of the phase diagram. The HF electrodynamics contains two diagrams, a vacuum bubble that renormalizes the chemical potential as $\hat{\mu}(z) \mapsto \hat{\mu}(z) + \delta\hat{\mu}(z)$ (the Hartree term):

$$\delta\hat{\mu}(z) \equiv q \sum_{\ell=1}^{N} \int \frac{kdk}{2\pi} \left(\psi_1^\dagger(E_\ell, k; z)\psi_1(E_\ell, k; z) + \psi_2^\dagger(E_\ell, k; z)\psi_2(E_\ell, k; z) \right) \tag{23}$$

and the exchange interaction (the Fock term). The explicit z-dependence of the Hartree correction is a gory reminder that the problem is solved in inhomogenous background. This is also the reason why already the Hartree correction is nontrivial: unlike the textbook situation where the shift $\delta\mu$ merely changes the numbers, here it is a radial function $\delta\mu(z)$ and its influence is also qualitative. So far, nobody even tried to do the whole HF calculation, and even just the Hartree term is not easy. We are plagued (1) by the UV divergences introduced by the modes close to $k = k_F$ which, as we have seen, peak most sharply near the boundary and can shatter the AdS space into pieces if not properly renormalized (2) by the IR divergences introduced by the modes with k close to zero, which extend far into large z values and can make the system unstable to forming a naked singularity.

Hard-wall Fermi liquid. The only case which is under good control is the hard-wall model of [12]: the UV divergences are resolved simply by not backreacting on the metric, i.e. solving just the Maxwell-Dirac system in fixed AdS metric (6) even without a black hole, and the IR divergences disappear by cutting off the space at some arbitrary z_0, so that we simply elliminate the IR region. The approximations are rather drastic, but they allow a complete solution. In pure AdS space, the solutions $\psi_{1,2}$ can be found analytically in terms of Bessel functions, the states form discrete and gapped bands, and we only have to solve the Maxwell equation (18). The outcome is given in Fig. 3. Hard wall acts as an infinite potential barrier, so the wavefunctions should die on it, and the condition $\psi_{1,2}(z_0) = 0$ determines the dispersion relation. The wall should not be charged, so in (19) the second term on the right-hand side equals zero, meaning that $A_t'(z_0) = 0$. The picture is that of a Fermi liquid, nicely filling the Fermi sea at momenta $k \leq k_F$ and having long-living quasiparticles. This model is an important starting point for more complicated setups, and has the advantage of being intuitive, but by itself is too simplistic. Indeed, we want to talk about hairy black holes, and here we don't even have one, as it is hidden behind the hard wall!

An attempt to study a simple setup but with a black hole was made in [16]. In this approach, we are limited to a single energy level, $\ell = 1$. This is justified only when the hair is just starting to form, right at the transition point. There is again no backreaction on metric, but the (fixed) metric is now taken to be the RN black hole. This is actually a big jump in difficulty: the wavefunctions oscillate near the horizon at any nonzero energy (Fig. 4(A)), so they can satisfy the IR boundary condition at any energy and momentum (we can always pick the phase so that $\psi'(z_h) = 0$), and the spectrum is continuous as there is no wall to create a gap. This is what forces us to consider the single-mode case: with the gapped hard-wall model we could add a *finite* number of modes, but now there is a *continuum* of them, N going to infinity even for arbitrarily small Q/q. The only way out is to *assume* there one mode only and solve the resulting Dirac-Maxwell system. This setup is convenient for understanding the transition itself, which turns out to be discontinuous (first-order) at finite temperature (Fig. 4(B), and likely infinite-order (Berezinskii-Kosterlitz-Thouless, BKT) at zero temperature, as we shall soon see.

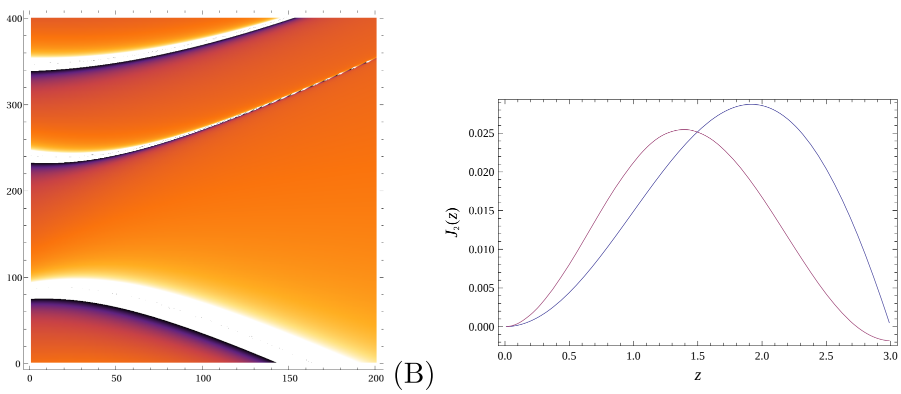
(A) 

Figure 3: (A) Dispersion relation $E_\ell(k)$ for the first two electron bands $\ell = 1, 2$ in hard-wall AdS space, for $\mu = 1, m = 1, q = 2$. The first band from bottom is the hole band, not an an electron band – its contribution can be absorbed in the redefinition of the parameters and it does not contribute to hair. The colormap shows the resolvent of the Dirac operator, $(D_z\Gamma^z + \vec{D}\cdot\vec{\Gamma} - m - E)^{-1}$, thus the bright white regions show the places where the resolvent diverges and a discrete bound state is formed. The horizontal axis is the momentum and the vertical axis the energy, both in computational units. (B) Wavefunctions $\psi_{1,2}$ (here for $\ell = 1$ and $k = 1$) are smooth everywhere - what happen exactly at the horizon we do not know in this model, as the space is cut off at $z = 3$.

Quantum electron star. The single-mode approach has taught us a lesson: already at the level of the gauge field only, the changes from the finite fermion density are drastic, and the resulting stress tensor is large at the horizon, so a change of the black hole metric is certainly expected. However, when we try to solve the Einstein equatioons, things become almost intractable. Both UV and IR divergences appear: the former because the currents diverge in continuous space, and the latter because the discrete bands fuse into a continuum in IR. The latter issue is most easily regularized by a hard wall, but a hard wall does not make much sense if we want to backreact on geometry. The regularization of the UV divergences is systematically discussed in [17, 18] and the bottom line is that there is a logarithmic short-distance divergence which can be regularized by point

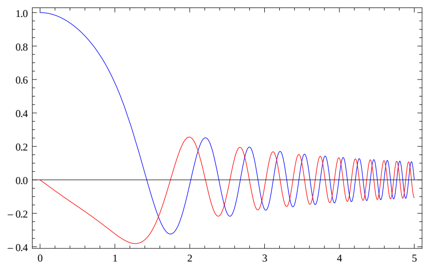
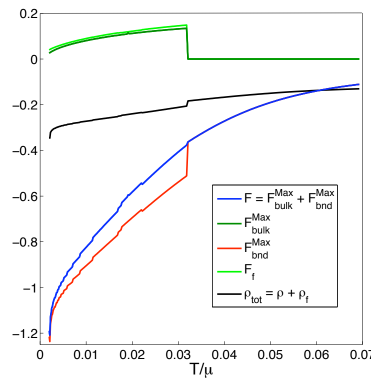
(A) 
(B) 

Figure 4: (A) Wavefunctions $\psi_{1,2}$in RN background, for $\ell = 1$ and $k = 1$, always oscillate and approach an essential singularity at the horizon, which indicates an instability: the metric changes and the degenerate horizon disappears. (B) The bulk action (or free energy F, from AdS/CFT correspondence) of the Maxwell (electric field), in blue, consists of the bulk and boundary contribution (dark green and red), the former practically identical to the contribution from fermions. All these are computed from the action (2-4). While the *total* free energy is continuous, it has a cusp, made manifest by the slight jump in density (black), a sign of first-order hair-forming transition.

splitting; in this procedure the cosmological constant becomes renormalized. This is not a drastic change: it will just change the numbers but not qualitative behavior. The IR problem is still unsolved. The approach of [18] is to put the system in global AdS space[4] whose radial slices are spheres, not planes, so the AdS radius provides a regulator. A perhaps more physical approach, motivated by consistent truncations from string theory, is to introduce a non-minimally coupled scalar, i.e., a dilaton that introduces a soft wall and suppresses the IR degrees of freedom in a continuous way, without an abrupt cutoff at some z_0, so the total bulk action is now

$$\begin{aligned} S_{\text{bulk}} &= \int d^4x\sqrt{-g}\left[R - V(\Phi) - \frac{1}{2}(\partial\Phi)^2 - \frac{Z(\Phi)}{4}\hat{F}^2\right] - \\ &- \int d^4x\sqrt{-g}\bar{\Psi}\left(\frac{1}{2}D_a\Gamma^a e^{\Phi} + \frac{1}{2}e^{\Phi}D_a\Gamma^a + m\right)\Psi, \end{aligned} \tag{24}$$

where the dilaton potential reproduces the AdS cosmological constant near the boundary, i.e., $\Phi(z \to 0) = 0$ and $V(\Phi \to 0) = 6, Z(\Phi \to 0) = 1$. It is not clear if one can ever remove the IR regulator. That is precisely the

[4]Dual field theory then lives on a sphere instad of a plane.

reason that we regard the dilaton regulator as more physical, since string theory constructions as a rule contain non-minimally coupled scalars, and the action (24) can be obtained by consistent truncation; whereas global AdS is essentially an ad hoc solution, though a very interesting one, with possible applications in AdS/condensed matter duality, where systems that live on surfaces (such as a sphere) appear naturally.

While this is still very much a work in progress,[5] preliminary results suggest that the RN-to-hairy-black-hole transition at zero temperature is an infinite-order (BKT) transition, where all derivatives of S remain smooth (Fig. 5). This is the point where the potential just starts deviating very slightly from the flat IR behavior in Fig. 1(A). At the end of this section we will try to understand this (still conjectural) numerical finding analytically.

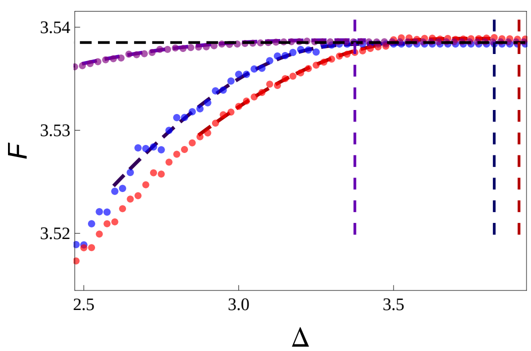


Figure 5: The bulk action (here denoted as free energy F, from AdS/CFT correspondence) as a function of the fermion mass (here denoted as $\Delta = 3/2 + m$) is very well fit by the BKT function $\exp(-c/\sqrt{\Delta_c - \Delta})$. The parameter c is determined by the chemical potential (we plot for three values $mu = 1.0, 1.5, 2.0$ in violet, blue, green). To the right of the transition point the action is independent of m as there is no hair, fermion density is zero, and so nothing depends on the fermion parameters. To the left of the transition point, the fermions form hair of nonzero density. Nobody knows yet how the near-horizon metric changes.

3.2. WKB star and electron star

WKB approach. We have followed the logical chain of reasoning from the point where the hair starts growing, having $Q/q \sim 1$ and deforming the black hole just a slight bit, towards larger and larger hair, eventually reaching the regime $Q/q \gg 1$. But this last regime is the easiest to approach, as the fermions become as close to classical as they can possibly be. A good starting point is the controlled expansion in $\hbar$, where we solve the

[5] With N. Chagnet, V. Djukić and K. Schalm.

Dirac equation in the eikonal approximation or, in other words, the WKB approach [15]. We express the wavefunction as

$$\psi_{1,2} = e^{\imath\theta_\pm}/\sqrt{p}, \quad p \equiv \sqrt{\hat{E}^2 - \hat{m}^2 - \hat{k}^2}, \tag{25}$$

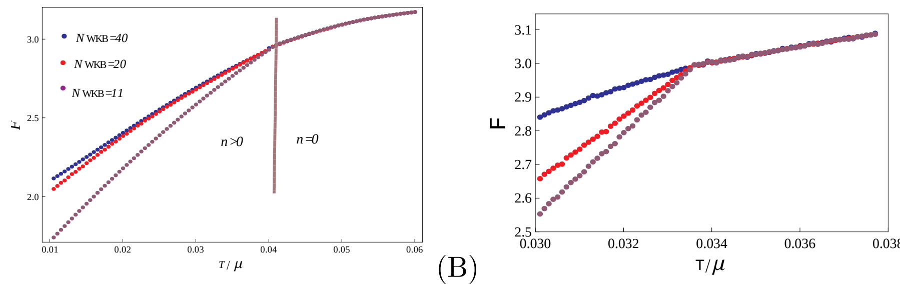
where p has the role of the canonical momentum. The wavefunction is nonzero between the turning points $z_\pm$, determined by the equation $p(z_\pm) = 0$. The explicit from of the phase $\theta_\pm$ as well as higher-order corrections to the phase can be found in [15], but the reader should in fact have no difficulty in deriving them, following the usual WKB procedure (though for the Dirac equation instead of the Schrödinger equation). Now the density and pressure are found by inserting the solution (25) into (22). The procedure can be iterated to obtain self-consistent solutions, but now we solve the whole system including the Einstein equations. It is instructive to plot the total on-shell action (2) as a function of temperature (remember that finite temperature is imposed through the corresponding boundary condition for the metric function F).[6] Fig. 6 plots the dependence $\mathcal{F}(T)$ in the vicinity of the transition value T_c: the derivative $\partial\mathcal{F}/\partial T$ undergoes a jump which is nothing but the entropy $\mathcal{S} \equiv \partial\mathcal{F}/\partial T$. We thus find a *first-order phase transition* at the point when Fermi hair starts forming. Of course, don't forget that the WKB approach is in fact *not* to be trusted very near the transition point: at the transition N changes from 0 to 1, which is far from the regime $N \ll 1$. But the qualitative insight that at finite temperature the system undergoes a non-symmetry-breaking transition is likely robust and we expect to prove it also within the more rigorous fully quantum-mechanical approach of the previous subsection. It is a hairy version of the celebrated Hawking-Page transition [19], and confirms the intuition that the hightmeperature phase is always a black hole; but now, the low-temperature phase is not simply a gas, but a dense fluid in AdS.

Ploting the density and pressure in Fig. 7(A), one finds that for high values of N they tend to a constant value in deep interior. This motivates the fluid ansatz taken in the electron star limit, now to be considered.

Electron star. Electron star is a charged, AdS version of the neutron stars, described as perfect fluid by the Oppenheimer-Volkov equations. The idea is to *assume* that the fermionic matter is a perfect fluid, and then express the energy density ρ, pressure p and charge density n in terms of *integrals* over energy and momenta (i.e., assume that the bound states are infinitely close, and the gaps between them vanish). The fluid approximation thus becomes exact in the limit of $N \to \infty$, as we expect from a semiclassical approximation. Anticipating the current and stress tensor of the form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \quad N_\mu = n u_\mu, \tag{26}$$

[6]In AdS/CFT, the bulk on-shell action $\mathcal{S}$ precisely equals the free energy $\mathcal{F}$ of the CFT side. But even without considering the details of the CFT, we can still make use of this interpretation to detect a phase transition in the system.



(A) (B)

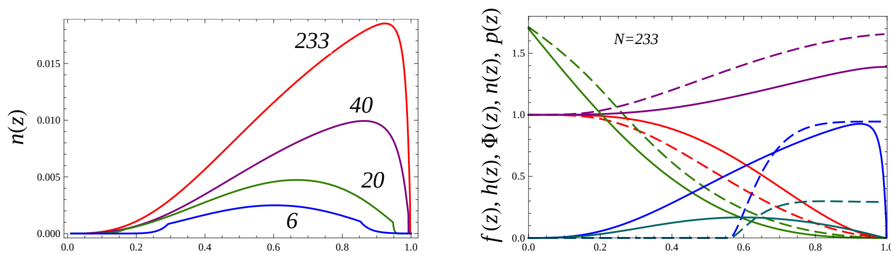
Figure 6: (A) The on-shell action or free energy as a function of temperature, in the presence of fermions. For low temperatures, the fermion density is finite and the derivative $\partial\mathcal{F}/\partial T$ jumps at $T = T_c$, a sign of first-order transition with the development of the hair. This is in line with the Dirac hair result in the previous figure, and indeed for the lowest number of levels N_{WKB} the transition is the sharpest. In (B) we zoom in into the transition region.

we can write the density starting from (22) and making use the optical theorem to relate it to the imaginary part of the Feynmann propagator G_F. This spells out as

$$\begin{aligned} \rho &= \int_0^{\hat{E}^2-k^2} \frac{dE}{2\pi} \int_0^{k_F} \frac{d^3k}{(2\pi)^3} \hat{E} \Im \mathrm{Tr} \imath \Gamma^0 G_F(E,k) \\ &= \int_0^{\hat{E}^2-k^2} dE \int \frac{k^2 dk}{4\pi^3} \frac{1}{2} \left(1 - \tanh\left(\frac{\beta}{2}\hat{E}\right)\right) \mathrm{Tr}(\imath\Gamma^0)^2 \delta\left(\hat{E} - \sqrt{k^2+m^2}\right) \\ &= \frac{1}{\pi^2} \int_m^{\hat{\mu}} dE E^2 \sqrt{E^2-m^2}. \end{aligned} \tag{27}$$

We similarly find the number density n, whereas the pressure need not be computed explicitly: since we work with an isotropic free Fermi fluid, its equation of state has to be $p = \rho - qn\hat{\mu}$. It is here that the approximate nature of the electron star with respect to the WKB star becomes obvious (Fig. 7): in WKB star there is an extra term in the pressure, coming from the nodes of the WKB wavefunction. One can check that the integral in (27) indeed approaches a constant as we go into deep interior. On the other hand, at some z_* when $\hat{\mu}(z_*) = m$ the density falls to zero: the star is a classical object and has a sharp border. So for $0 < z < z_*$ we continue the metric to the RN metric (the metric outside a charged isotropic object).

Since we can express n, ρ, p explicitly, we get a nice system of local ordinary differential equations in F, G, A_t, with all quantum expectation values pulled under the rug. This completes the circle, and brings another universal message: *due to Pauli principle, fermionic operators are never local, except in two extreme cases: when only one state is occupied (so the format of the Slater determinant is* 1×1*, i.e., it contains a single state), or* *when infinitely many states are occupied, so the Slater determinant turns into a classical, continuous probability density.* In Fig. 7 we can see how the WKB solution captures the quantum "tails" near the turning points, which the electron star does not have. It is also instructive to compare this solution to the Oppenheimer-Volkov equations in flat space: in the latter case, $\hat{m} \sim 1/\sqrt{F}$ is always larger than $\hat{\mu} \sim 1/F\sqrt{G}$, unlike in AdS where $\hat{m} \sim 1/z\sqrt{F}$ and for $z > z_*$ it becomes smaller than the local chemical potential, so the integral in (27) has a nonzero range. This is because AdS acts like a potential box that can hold the charged fermions together against electrostatic repulsion. In flat space that does not happen, and we have only neutron stars, not electron stars.



(A) (B)

Figure 7: (A) Density of the finite temperature WKB star at various fillings N_{WKB}; besides the classically allowed region, there are also exponentially decaying tails in the classically forbidden region, where $V_{\text{eff}} > 0$. (B) In the electron star (fluid limit), there are no such tails and the star has a sharp border. Taken over from [15]. (B) Comparison of the WKB solution (full lines) and the electron star solution (dashed lines) at the same chemical potential, fermion charge and mass. We plot the metric functions f, h (F, G in the main text) in red and violet, the gauge field Φ (A_t in the main text) in green, and density and pressure n, p in blue and dark green. The metric solutions do not differ much, despite the long quantum WKB tails, absent in the electron star.

3.3. Lifshitz metric, BKT transition and the missing pieces

In the framework of the electron star model, the Einstein-Maxwell equations can be solved analytically, thanks to the fact that, in deep IR, $n, \rho, p =$ const. and we can employ a scaling ansatz for the metric. The idea is to match the IR expansion around the scaling solution to the UV expansion around pure AdS. With ansatz of the form $g_{tt} \propto -1/z^\alpha$, $g_{ii} \propto 1/z^\beta$ and $g_{zz} = 1/z^2$ (one metric component we can fix at will as it amounts to picking the gauge for the metric), equations of motion give the IR solution

$$ds^2 = -\frac{1}{z^{2\zeta}}dt^2 + \frac{1}{z^2}d\vec{x}^2 + \frac{1}{z^\zeta}dz^2$$

$$A_t = \frac{1}{z^\zeta}, \quad \mathcal{L} = \frac{2}{z^{2\zeta}} + 6 - \partial_z A_t^2 - nA_t^2 - p_\perp, \tag{28}$$

where in the expression for the total Lagrangian density in the second line, we have inserted in the action (2) the solutions for the metric and the gauge field, as well as the constant (z-independent) solution obtained for ρ in (27) and similarly for n, p. Three important conclusions can be drawn: (1) the IR metric is scale-invariant, with anisotropic scaling of time and space, so that the scaling transformation has the form $t \mapsto \lambda t$, $\vec{x} \mapsto \vec{x}\lambda^{1/\zeta}$ (2) the on-shell Lagrangian density effectively describes a *massive vector field*, with mass squared equal to fermion density n (3) the fermionic contribution to the action equals the pressure. The second point agrees with the known result that Lifshitz black holes are generated by Proca fields [21], and what happens is the Abelian-Higgs mechanism: fermion density acquires a finite expectation value which in turn breaks the $U(1)$ symmetry, giving the photon a mass. The third point is expected within a fluid model, since the action of an ideal Lorentz-invariant (semi)classical fluid equals its pressure [7]. In the fluid limit we can also understand the first-order transition at finite temperature, because it is just a van der Waals-type liquid-gas transition.

We have seen that the thermal transition from RN to a Lifshitz black hole is of first order, and that the $T = 0$ transition is apparently a BKT (infinite order) transition. The latter is not quite clear yet because, as we have emphasized, nobody has yet managed to peek into the deep IR, it remains hidden behind the hard wall. But if we tentatively accept the numerical evidence for the infinite-order transition, can we understand it theoretically? The key lies in understanding how the AdS_2 throat disappears. The conformality-breaking mechanism of [22, 23] gives an idea, though the details are still missing. The crucial moment is that the near-horizon geometry is AdS_2. Right at the horizon ($s \to -\infty$) the potential is approximately constant. In the UV of the AdS_2 throat, which is around some finite value s_0, the potential behaves as $-c/(s - s_0)^2$. This inverse-square potential is known to describe conformal quantum mechanics when $c > -1/4$. For $c = -1/4$ the conformal invariance breaks. discrete states appear and the effective potential is not consistent unless regularized as

$$V_{\mathrm{eff}} = \frac{c}{(s - s_0)^2} - v\delta(s - s_0), \tag{29}$$

and the solution of the effective Schrödinger equation is

$$\psi(r) = c_+(s - s_0)^{\alpha_+} + c_-(s - s_0)^{\alpha_-}, \quad \alpha_\pm = \frac{1}{2} \pm \sqrt{c + \frac{1}{4}}, \tag{30}$$

and the ratio c_+/c_- is given in terms of Bessel functions $J_{1/2}$ and $J_{-1/2}$:

$$\frac{c_+}{c_-} = -\epsilon^{\alpha_- - \alpha_+} \frac{\gamma + \alpha_-}{\gamma + \alpha_+}, \quad \gamma = \sqrt{v} \frac{J_{1/2}(\sqrt{v})}{J_{-1/2}(\sqrt{v})} \tag{31}$$

The solution (30) diverges at $s = s_0$ unless we introduce a cutoff at some distance ϵ from s_0. Imposing the renormalization condition that c_+/c_- remains independent of ϵ, we get the β-function of the renormalization group as (ℓ being the RG scale):

$$\beta \equiv \frac{d\gamma}{d\ell} = (c + 1/4) - (\gamma + 1/2)^2. \tag{32}$$

And we're done: the fixed points of the above flow equation are easily found to be $-\alpha_\mp$. For $\gamma = -\alpha_\mp$ we get the solution for ψ from (30) with $c_\pm = 0$ respectively. The free energy scaling is obtained as $S_{\mathrm{on-shell}} = \mathcal{F} \propto \int d\ell/\beta$, which gives just the form found in Fig. 5. However, the presence of both a hard-wall cutoff in z and the soft-wall dilaton, completely unaccounted for in the above analysis, clearly suggest more work is needed for everything to click together.

4. Wormholes with fermion hair

The lengthy review we have given so far is meant to be self-contained and helpful for those interested in understanding and contributing to the problem of black hole instabilities with fermionic matter. As we have seen, it contains some puzzling questions and is of more than technical interest (after all, the whole field has been active mostly for the last fifteen years or so). But we also want to point out that with the methodological powerhouse of the HF, WKB and fluid methods, one can tackle new problems. A recent issue where fermions at finite density seem very relevant is the search for traversable wormholes.

The motivation for this story lies mainly in the celebrated black hole information paradox: as far as we know, the Hawking radiation is thermalized, meaning that the information content of the matter falling into the black hole is lost. A possible way out or, at least, a way to better understand the issue, is to consider the maximally extended Carter-Penrose diagram of a black hole, which contains two horizons and two spacetimes. If transport between the two were possible, one could imagine that the information is not lost because the matter falling into one horizon is entangled with the matter on the opposite side. This is the idea of the ER=EPR conjecture [24]. In order to build a traversable wormhole, one needs negative that the stress-energy tensor averaged over a geodesic be negative, thus violating the so-called averaged negative energy condition (ANEC) [25, 26]. This will never happen with conventional classical matter. One needs either exotic fields or quantum corrections. Recently however, a few traversable wormholes have been realized with only standard-model matter. The most "conservative" is the setup of [27] which creates negative energy by considering a particle-hole symmetric spectrum of massless fermions in a mangetic monopole field: because of the negative Landau levels, the net energy is negative. The starting point is thus a pair of magnetically charged RN black holes with magnetic charges H and $-H$, with the hope that the negative energy Landau levels will push the averaged stress tensor to large enough absolute values to open up a wormhole. In this way, [27] constructs a quasi-stationary (long-living) wormhole in *assymptotically flat* space. In AdS, negative energy density can easily be constructed by coupling the two boundaries nonlocally: in this way temporary wormholes, opening up for the finite duration of the pertrubation, can be constructed [28], and even eternal wormholes are possible but at the cost of much more exotic boundary CFTs and their couplings [29, 30, 31]. Here we are interested in making a wormhole in a more "down-to-earth" manner, by growing negative-energy fermion levels as in [27]. The task is to make such wormholes more stable, and to see if they survive at higher fermion density rather than just a single wavefunction as in [27]. Here the previously develped methods can help us.

Magnetic electron star. The crucial consequence of the magnetic field is the Landau quantization. The motion along the x-coordinate is quantized into discrete levels, whereas the motion along y is not quantized and introduces degeneracy. The quantization along x-axis makes our life somewhat easier – even without any IR cutoff the ground state wavefunction now has a discrete quantum number, the Landau level m_j. The magnetic field breaks the spherical symmetry of the wavefunctions down to cylindrical, so it is convenient to introduce the polar angles θ, ϕ:

$$ds^2 = -A(z)dt^2 + B(z)dz^2 + C(z)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{33}$$

and to pick a different gamma matrix basis: $\Gamma^0 = \imath\sigma_1 \otimes \hat{1}$, $\Gamma^1 = \sigma_2 \otimes \hat{1}$, $\Gamma^2 = \sigma_3 \otimes \sigma_1$, $\Gamma^3 = \sigma_3 \otimes \sigma_2$. Separating the variables and representing the wavefunction as

$$\Psi = \sum_{m_j=-j}^{j} \left(\psi_+\left(m_j; z\right), \psi_-\left(m_j; z\right)\right) \otimes \left(\eta_1\left(m_j; \theta\right), \eta_3\left(m_j; \theta\right)\right) e^{\imath m_j \phi}, \tag{34}$$

where j is the total number of Landau levels $j = (H-1)/2$, we get the fully spin-polarized solution ($\eta_2 = 0$) for zero fermion mass:

$$\psi_\pm(m_j; z) = \exp\left(\pm\imath E(m_j)\int_0^z dz'\sqrt{\frac{B(z')}{A(z')}}\right), \tag{35}$$

$$\eta_1(m_j; \theta) = \frac{e^{\imath H\sin\theta/2}}{\sqrt{\sin\theta}}\left(\tan\frac{\theta}{2}\right)^{m_j}.$$

For nonzero mass, we can perform a Foldy-Wouthuysen transform starting from the above solution. Unlike the massless case considered in [27], the resulting stress-energy tensor will not be traceless, but that is precisely what will guve us extra stability. The reason this is consistent is the Landau quantization: the levels for different m_j are gapped from each other and each Landau level can be treated as a single-particle solution which does not mix with other Landau levels. This results in the stress tensor

$$\langle T_{zz} \rangle = \frac{E_n}{(1+z^2)^2} \left(\sin 2\alpha - \cos 2\alpha \right), \quad \tan \alpha = -m/E(m_j). \tag{36}$$

Fig. 8(A) shows the radial pressure T_{rr} as a function of energy, the outcome being that *positive* stress energy tensor is produced for $0 > E > -m$. In order to avoid this positive contribution, the Landau level spacing has to be large enough, i.e., larger than the mass gap (at zero mass this condition is trivially satisfied, as it simply means that any finite $E(m_j = 1)$ will do; this is the case studied in [27]). The simplest gapping mechanism we can think of is the chemical potential, i.e. an electrostatic field in addition to the magnetostatic one. The black hole thus has to become dyonic, with magnetic charge H and electric charge e. Assuming we have ensured the negativity of (36), we can write it in the form $T_{zz} = -\tau/(1+z^2)^2$, with τ a positive constant. Its magnitude roughly determines the size of the wormhole opening.

(A) (B)

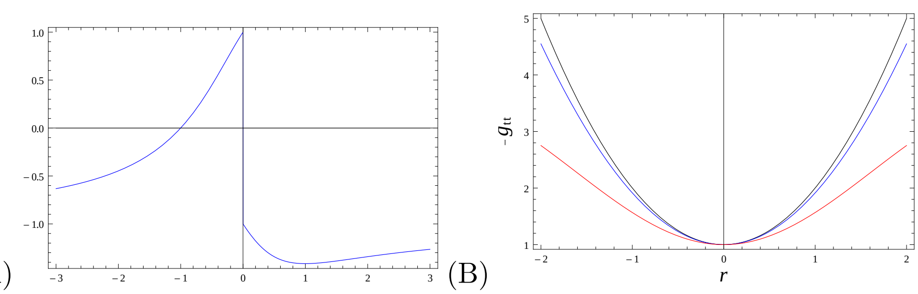


Figure 8: (A) Radial component of the stress-energy tensor $\langle T_{rr} \rangle$ as a function of the (discrete) fermion energy E. Positive contribution only comes when $> E(m_j) > -m$. In order to avoid this range of energies we need a nonzero chemical potential (i.e., electric field, resulting in a dyonic black hole) to stabilize the wormhole with massive fermionic hair. (B) The solution for the metric component g_{tt} in the intermediate region, as a function of the radial coordinate r, for $\tau = 0, 0.05, 0.10$ (black, blue, red). Wormhole solutions (blue, red) are quantitatively very close to the unperturbed black hole (black) but qualitativrly different as there is no zero anymore.

Wormhole solution and matching. Having computed the stress-energy tensor (36), we can solve the Einstein equations. The strategy is again matching the expansions, but now we have three regions: the far region which is asymptotically AdS or even flat (we have mentioned that in the presence of magnetic field discrete bound states can form even in absence of AdS boundary), the intemediate region is a slightly perturbed near-horizon AdS_2 region of our magnetic RN geometry, and the inner region, the wormhole throat that opens up, turns out to be a *global* AdS_2 at leading order, so it has a spherical boundary continuing onto the intermediate regions. The inner, near-global-AdS$_2$ metric in the form (33) at leading order reads

$$A(z) = R_0^2 \left[1 + z^2 - 8\pi\tau \left(z^2 + \left(3z + z^3\right) \arctan z - \log\left(1 + z^2\right)\right)\right]$$
$$B(z) = R_0^4/A(z), \quad C(z) = R_0^2 \left[1 + 8\pi\tau \left(1 + z \arctan z\right)\right]. \tag{37}$$

This solution is to be matched to the intermediate-region solution. Now large z corresponds to the wormhole mouth, i.e., the matching is to be done at large z, where small z is the "center" of the wormhole throat. The solution to match onto is the RN black hole metric:

$$ds^2 = -l^2 f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$
$$l = \frac{R_0}{2\pi^2\tau}, \quad R = \frac{r - \sqrt{\pi}\sqrt{e^2 + H^2}}{2\pi^2\tau}. \tag{38}$$

The solution thus exists for any choice of e and H. But for large e (in other words, for a large chemical potential), the density of the hair will increase significantly and we should repeat the WKB star or electron star approach. in AdS this is a simple matter, proving the stability of the configuration even at high densities. The interesting question is, can it work also in assymptotically flat space? In absence of magnetic field, the answer is certainly no – without an AdS boundary, there is nothing to equilibrate the electrostatic repulsion of electrons. But in the presence of magnetic field, one might obtain a stable charged hairy wormhole if the change in the near-horizon geometry is sufficient to effectively decrease the electrostatic energy density. This is the logical immediate task for future work.

We finish this short review of our work in progress on hairy wormholes with a somewhat more ambitious task. The dyonic wormhole model considered here is obviously quite simplistic and artificial. A much more realistic model is to start from a pair of Kerr black holes and see if these can open up a wormhole in a manner analogous to the scenario we have considered. In this case the magnetic field would be generated self-consistently by the (rotating) fermionic hair, removing the need for the magnetic monopole charge. Such an object would come much closer to realistic astrophysical matter.

5. Instead of a conclusion

We have given a crack and practical review of the insights and technologies needed to describe and understand hairy black holes in anti-de Sitter space. The phase diagram in the presence of nonzero fermion density is quite rich, and it involves two deep and universal phenomena. First, the finite-temperature hairy black holes develop through a discontinuous phase transition akin to the Hawking-Page transition (indeed, it is precisely the Hawking-Page transition but at finite density). The standard lore that at high enough temperatures black holes will always form is confirmed. Notice this is true at any fermion mass and charge, and thus at any occupation number, from a single wavefunction to the fluid limit, so the finding is definitely robust. Second, at zero temperature the transition is driven by the fermionic charge and/or chemical potential, i.e., electric charge of the black hole. In this case the black hole vanishes infinitely slowly, in a BKT transition that can be understood as the breaking of the one-dimensional conformal symmetry of the wavefunctions in the effectve inverse-square potential well. This is solely the consequence of the near-horizon physics, independent of the AdS boundary. Similar conformality-breaking infinite-order transitions are known in various backgrounds in string theory. Maybe one could relate the case described here to some consistent top-down model.

As mentioned in the Introduction, we have deliberately left out extensions and applications of the formalism described, for reasons of space and also generality of discussion. The field of applications closest to our experience is the AdS/CFT correspondence. Electrically charged black holes are dual to field theories at finite $U(1)$ density. The transition from a bald black hole to a hairy black hole is thus a transition between two phases at equal chemical potential. How do they differ then? We know that a black hole is dual to the Coulomb (deconfined) phase of some non-Abelian finite-temperature gauge theory [1, 4]; in the simplest setup coming from type IIB string theory, it is the $\mathcal{N} = 4$ supersymmetric $SU(N)$ theory. Coulomb phase means that the $U(1)$ charge is carried by $SU(N)$-gauge-charged operators, in our case fermions ("mesinos") and thus not visible to low-energy probes, since at low energies all operators are likely $SU(N)$-gauge-neutral. The hairy phase describes a dual field theory where the charge is carried by gauge-neutral operators ("baryons") and thus visible to probes such as a photon. This viewpoint was tried and confirmed in [11, 12, 15, 23]. It has realizations in condensed matter systems such as strange metals and heavy fermion materials. In this case, the gauge fields are emergent and arise from the spin-charge separation, and the transition between a black hole and a hairy geometry is a transition between a non-Fermi liquid, where most of the charge is carried by complicated excitatons that are nor directly seen in the spectrum, and a Fermi liquid where the fundamental degrees of freedom are just renormalized electrons. In QCD, this picture describes the phase diagram at intermediate energy scales and finite densities, where a black hole describes quark-gluon plasma, and a hairy solution describes either the color condensate or conventional barionic matter depending on the details of the model. One can learn a lot on AdS/condensed matter and AdS/QCD from [5, 6].

Finally, the search for wormhole solutions and how fermionic hair might stabilize them is likely to become very important in the future, in connection to the quantum information theory and the firewall, ER=EPR and other approaches to the black hole information problem. One can use much of the formalism developed for hairy black holes, but the interpretation is still challanging. It is also unclear how realistic the wormhole proposal is if we work with only conventional, standard model matter, i.e. is it just an important proof of concept or a realistic model?

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