Courant and Roytenberg bracket and their relation via T-duality *

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Abstract

We consider the σ-model for closed bosonic string propagating in the coordinate dependent metric and Kalb-Ramond field. Firstly, we consider the generator of both general coordinate and local gauge transformations. The Poisson bracket algebra of these generators is obtained and we see that it gives rise to the Courant bracket. Secondly, we consider generators in a new basis, consisting of the σ derivative of coordinates, as well as the auxiliary currents. Their Poisson bracket algebra gives rise to the Courant bracket, twisted by the Kalb-Ramond field. Finally, we calculate the algebra of the T-dual generator. The Poisson bracket algebra of T-dual generator gives rise to the Roytenberg bracket, equivalent to the bracket obtained by twisting the Courant bracket by the non-commutativity parameter, which is T-dual to the Kalb-Ramond field. We show that the twisted Courant bracket and the Roytenberg bracket are mutually related via T-duality.

1. Bosonic string action

The closed bosonic string is moving in a curved background, associated with the symmetric metric tensor field $G_{\mu\nu} = G_{\nu\mu}$, the antisymmetric Kalb-Ramond tensor field $B_{\mu\nu} = -B_{\nu\mu}$, as well as the scalar dilaton field Φ. If we consider the case of constant dilation field, in a conformal gauge

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$g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$, the action is given by [1]

$$S = \int_\Sigma d^2\xi \mathcal{L} = \kappa \int_\Sigma d^2\xi \Big[\frac{1}{2}\eta^{\alpha\beta}G_{\mu\nu}(x) + \epsilon^{\alpha\beta}B_{\mu\nu}(x)\Big]\partial_\alpha x^\mu \partial_\beta x^\nu . \tag{1}$$

The integration goes over two-dimensional world-sheet Σ parametrized by $\xi^\alpha(\xi^0 = \tau, \xi^1 = \sigma)$. Coordinates of the D-dimensional space-time are $x^\mu(\xi)$, $\mu = 0, 1, ..., D-1$, $\epsilon^{01} = -1$ and $\kappa = \frac{1}{2\pi\alpha'}$. In a usual way, we derive the expression for the canonical momenta

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \kappa G_{\mu\nu}(x)\dot{x}^\nu - 2\kappa B_{\mu\nu}(x)x'^\nu, \tag{2}$$

and obtain the Hamiltonian

$$\mathcal{H}_C = \frac{1}{2\kappa}\pi_\mu (G^{-1})^{\mu\nu}\pi_\nu - 2x'^\mu B_{\mu\nu}(G^{-1})^{\nu\rho}\pi_\rho + \frac{\kappa}{2}x'^\mu G^E_{\mu\nu}x'^\nu , \tag{3}$$

where $G^E_{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}$ is the effective metric. It is possible to rewrite the Hamiltonian in a more convenient form

$$\mathcal{H}_C = \frac{1}{4\kappa}(G^{-1})^{\mu\nu}\Big[j_{+\mu}j_{+\nu} + j_{-\mu}j_{-\nu}\Big] . \tag{4}$$

where the currents $j_{\pm\mu}$ are defined by

$$j_{\pm\mu}(x) = j_{0\mu} \pm j_{1\mu} = \pi_\mu + 2\kappa \Pi_{\pm\mu\nu}x'^\nu , \tag{5}$$

and $\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2}G_{\mu\nu}$ are composite fields. The $\tau-$component of current j_μ will be marked as an auxiliary current $j_{0\mu} = \pi_\mu + 2\kappa B_{\mu\nu}x'^\nu \equiv i_\mu$. Now, we rewrite the equation (5)

$$j_{\pm\mu} = i_\mu \pm \kappa G_{\mu\nu}x'^\nu . \tag{6}$$

The generalized current is defined by

$$J_C(\xi, \Lambda^C) = \xi^\mu(x)i_\mu + \Lambda^C_\mu(x)\kappa x'^\mu , \tag{7}$$

where ξ^μ and Λ^C_μ are some coordinate dependent parameters. Due to their index structure, the former parameters can be regarded as vector field components and the latter as 1-form components. The suitable language for description of generalized currents is the one of generalized geometry [2]. The generalized vectors are direct sum of elements of tangent and cotangent bundle over some manifold, meaning that the generalized geometry treats vectors and 1-forms on equal footing. Therefore, it is possible to consider the generalized current as the function on generalized vector $\xi \oplus \Lambda^C$.

Besides the algebra of these generalized currents, we are also interested in algebra of T-dual generalized currents. T-duality [1, 3] is inherent to string theory and represents the equivalence of two seemingly different physical theories. The equivalence manifests itself in a way that all quantities in one theory are identified with quantities in its T-dual theory. It was firstly noticed in case of the bosonic closed string with one dimension compactified to a radius R. In that case, mass spectrum is given by [1]

$$M^2 = \frac{K^2}{R^2} + W^2 \frac{R^2}{\alpha'^2}\,, \tag{8}$$

where K is momentum and W winding number. It is obvious that the mass spectrum remains invariant upon simultaneously swapping $K \leftrightarrow W$ and $R \leftrightarrow \frac{\alpha'}{R}$. What can be concluded is that spectrums of two theories that both have one dimension compactified to a circle, where in one case the circle is of small and in the other of large radius, are indistinguishable. The momenta in one theory are winding numbers in its T-dual theory, and vice versa.

We consider the T-duality realized without changing the phase space [4]. Its transformation rules between the canonical variables are given by

$$\pi_\mu \cong \kappa x'^\mu\,. \tag{9}$$

When the above relation is integrated over space parameter σ, we obtain that the T-duality interchanges momenta with the winding numbers. The background fields have their T-dual counterparts as well [5], namely

$$^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}, \quad ^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}\,, \tag{10}$$

where $\theta^{\mu\nu} = -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}$ is the non-commutativity parameter. Two variables are said to be mutually T-dual if they transform one into another when simultaneously both (9) and (10) are applied [4].

Now we follow the same procedure as above for constructing the T-dual generalized currents. Applying (9) and (10) to (5), we obtain the current $l^\mu_\pm$, T-dual to current $j_{\pm\mu}$

$$l^\mu_\pm = k^\mu \pm (G_E^{-1})^{\mu\nu}\pi_\nu, \quad k^\mu = \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu\,. \tag{11}$$

The Hamiltonian can be expressed in terms of these currents by

$$\mathcal{H}_C = \frac{1}{4\kappa} G^E_{\mu\nu}\Big(l^\mu_+ l^\nu_+ + l^\mu_- l^\nu_-\Big)\,. \tag{12}$$

Substituting the expression (11) in (12), we obtain the Hamiltonian in the form (4). The Hamiltonian remains invariant under T-duality.

The generalization of current $l^\mu_\pm$ is given by

$$J_R(\xi_R, \Lambda) = \xi^\mu_R(x)\pi_\mu + \Lambda_\mu(x)k^\mu\,. \tag{13}$$

Instead of auxiliary current i_μ and coordinate $\sigma-$derivative x'^μ, the basis for T-dual generalized currents consist of T-dual auxiliary current k^μ and canonical momenta π_μ. The coefficients ξ^μ_R are arbitrary vector field components, while Λ_μ are arbitrary 1-form components. Next, we will see how these currents are related to the symmetry generators.

2. Symmetry generators algebra

Let us start with the general coordinate transformations. They are generated by

$$\mathcal{G}_{GCT}(\xi) = \int_0^{2\pi} d\sigma \xi^\mu(x)\pi_\mu \,. \tag{14}$$

Action of the general coordinate transformation on background fields is governed by the action of Lie derivative [6]

$$\delta_\xi G_{\mu\nu} = \mathcal{L}_\xi G_{\mu\nu},\ \delta_\xi B_{\mu\nu} = \mathcal{L}_\xi B_{\mu\nu} \,. \tag{15}$$

The Lie derivative is defined by $\mathcal{L}_\xi = i_\xi d + d i_\xi$, where the interior product i_ξ acts as a contraction with a vector field ξ, while the exterior derivative d extends the concept of the differential of a function to differential forms. The commutator of two Lie derivatives results in another Lie derivative

$$\mathcal{L}_{\xi_1}\mathcal{L}_{\xi_2} - \mathcal{L}_{\xi_2}\mathcal{L}_{\xi_1} = \mathcal{L}_{[\xi_1,\xi_2]_L} \,, \tag{16}$$

where by $[\xi_1, \xi_2]_L$, we marked the Lie bracket between two vector fields ξ_1 and ξ_2. Explicitly, it is given by

$$([\xi_1, \xi_2]_L)^\mu = \xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu \,. \tag{17}$$

Calculating the Poisson bracket algebra of general coordinate transformations generators $\mathcal{G}_{GCT}$ (14), we obtain the relation:

$$\{\mathcal{G}_{GCT}(\xi_1),\, \mathcal{G}_{GCT}(\xi_2)\} = -\mathcal{G}_{GCT}([\xi_1, \xi_2]_L) \,. \tag{18}$$

We note that this algebra gives rise to Lie bracket.

We would like to extend the generator G_{GCT} so that it includes the local gauge transformations as well. They are generated by [6]

$$\mathcal{G}_{LGT}(\Lambda) = \int d\sigma \Lambda_\mu \kappa x'^\mu \,, \tag{19}$$

where Λ_μ are gauge parameters that are 1-form components. The action of local gauge transformations on background fields is given by

$$\delta_\Lambda G_{\mu\nu} = 0,\ \delta_\Lambda B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \,. \tag{20}$$

When gauge parameter is changed so that total derivative of arbitrary function is added $\Lambda_\mu \to \Lambda_\mu + \partial_\mu \lambda$, the action of generator does not change.

Hence, different gauge parameters generate same symmetry and therefore the generator is reducible.

Local gauge transformations are T-dual to general coordinate transformations, due to relation (9). The sum of both generators (14) and (19) results in symmetry generator that is T-dual to itself

$$\mathcal{G}(\xi \oplus \Lambda) = \int d\sigma \Big[\xi^\mu(x)\pi_\mu + \Lambda_\mu \kappa x'^\mu\Big] . \tag{21}$$

Using the language of generalized geometry, we treat the generator (21) as a function of generalized gauge parameter, defined as the direct sum of vector and 1-form parameter $\xi \oplus \Lambda$.

2.1. Courant bracket

The Poisson bracket algebra of the generators (21) is given by

$$\{\mathcal{G}(\xi_1 \oplus \Lambda_1), \mathcal{G}(\xi_2 \oplus \Lambda_2)\} = -\mathcal{G}([\xi_1 \oplus \Lambda_1, \xi_2 \oplus \Lambda_2]_C) , \tag{22}$$

where $[\xi_1 \oplus \Lambda_1, \xi_2 \oplus \Lambda_2]_C$ is the Courant bracket [7]. It is considered to be the generalized geometry extension of the Lie bracket. Just like the Lie bracket acts on vectors, Courant bracket acts on generalized vectors. Its full expression is given by

$$[\xi_1 \oplus \Lambda_1, \xi_2 \oplus \Lambda_2]_C = [\xi_1, \xi_2]_L \oplus \Big(\mathcal{L}_{\xi_1}\Lambda_2 - \mathcal{L}_{\xi_2}\Lambda_1 - \frac{1}{2} d(i_{\xi_1}\Lambda_2 - i_{\xi_2}\Lambda_1)\Big) . \tag{23}$$

The first term $[\xi_1, \xi_2]_L$ on the right hand side of previous equation gives the vector, while the other terms give 1-form. It has been shown before [8] that the algebra of symmetry transformations gives rise to the Courant bracket.

Courant bracket cannot be a bracket of a Lie algebra, as it does not satisfy the Jacobi identity. However, this does not pose a problem, as the deviation from Jacobi identity is closed form that disappears after the integration and hence correspond to a trivial symmetry. The Jacobiator of Courant bracket is given by [2]

$$[[\xi_1 \oplus \Lambda_1, \xi_2 \oplus \Lambda_2]_C, \xi_3 \oplus \Lambda_3]_C + cyclic = dNij(\xi_1 \oplus \Lambda_1, \xi_2 \oplus \Lambda_2, \xi_3 \oplus \Lambda_3)_C , \tag{24}$$

where the Nijenhuis operator is defined by

$$Nij(\xi_1 \oplus \Lambda_1, \xi_2 \oplus \Lambda_2, \xi_3 \oplus \Lambda_3)_C = \frac{1}{3} < (\xi_1 \oplus \Lambda_1, \xi_2 \oplus \Lambda_2)_C, \xi_3 \oplus \Lambda_3 > + cycl. \tag{25}$$

and $< \xi_1 \oplus \Lambda_1, \xi_2 \oplus \Lambda_2 >= \frac{1}{2}(\xi_1(\Lambda_2) - \xi_2(\Lambda_1))$ is the natural inner product between generalized vectors.

2.2. Twisted Courant bracket

We claimed that it is possible to obtain the generalized currents from the symmetry generators. To demonstrate that, let us now define the new gauge parameter

$$\Lambda_\mu^C = \Lambda_\mu + 2B_{\mu\nu}\xi^\nu \,. \tag{26}$$

This change of parameter can be interpreted as the B-transformation acting on the generalized gauge parameter

$$\mathcal{O}_B\Lambda = \begin{pmatrix} \delta_\nu^\mu & 0 \\ 2B_{\mu\nu} & \delta_\mu^\nu \end{pmatrix} \cdot \begin{pmatrix} \xi^\nu \\ \Lambda_\nu \end{pmatrix} = \begin{pmatrix} \xi^\mu \\ \Lambda_\mu^C \end{pmatrix} \,. \tag{27}$$

It is suitable to rewrite the generator (21) in the new basis

$$\mathcal{G}_{\mathcal{C}}(\xi \oplus \Lambda^C) = \int d\sigma \Big[\xi^\mu i_\mu + \kappa \Lambda_\mu^C x'^\mu\Big] \,. \tag{28}$$

Comparing the expression for the symmetry generator $\mathcal{G}_{\mathcal{C}}$ (28) with the expression for generalized currents (7), we see that this generator is the charge corresponding to the current J_C.

The change of basis results in the appearance of H-flux in the basis algebra

$$\{i_\mu(\sigma), i_\nu(\bar\sigma)\} = -2\kappa B_{\mu\nu\rho} x'^\rho \delta(\sigma - \bar\sigma) \,, \tag{29}$$

where

$$B_{\mu\nu\rho} = (dB)_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \tag{30}$$

is the H-flux. The Poisson bracket algebra of generators was obtained in [4, 9] and written in the form

$$\{\mathcal{G}_{\mathcal{C}}(\xi_1 \oplus \Lambda_1^C), \, \mathcal{G}_{\mathcal{C}}(\xi_2 \oplus \Lambda_2^C)\} = -\mathcal{G}_{\mathcal{C}}([\xi_1 \oplus \Lambda_1^C, \xi_2 \oplus \Lambda_2^C]_B) \,. \tag{31}$$

The bracket $[\xi_1 \oplus \Lambda_1^C, \xi_2 \oplus \Lambda_2^C]_B$ is called the Courant bracket, twisted by a 2-form $2B_{\mu\nu}$. This twist is realized by

$$[\mathcal{O}_B(\xi_1 \oplus \Lambda_1), \mathcal{O}_B(\xi_2 \oplus \Lambda_2)]_C - \mathcal{O}_B[\xi_1 \oplus \Lambda_1, \xi_2 \oplus \Lambda_2]_C = H(\xi_1, \xi_2, .) \,. \tag{32}$$

We see that it differs from the Courant bracket (23) by the contraction of H-flux $H = dB$ (30) with two gauge parameters ξ_1 and ξ_2. The full expression for this bracket is given by

$$\begin{aligned}[\xi_1 \oplus \Lambda_1^C, \xi_2 \oplus \Lambda_2^C]_B = [\xi_1, \xi_2]_L \oplus \Big(&\mathcal{L}_{\xi_1}\Lambda_2^C - \mathcal{L}_{\xi_2}\Lambda_1^C \\ &- \frac{1}{2}d(i_{\xi_1}\Lambda_2^C - i_{\xi_2}\Lambda_1^C) + H(\xi_1, \xi_2, .)\Big) \,.\end{aligned} \tag{33}$$

2.3. Roytenberg bracket

Finally, we are going to consider one more transformation of the gauge parameter

$$\xi_R^\mu = \xi^\mu + \kappa\theta^{\mu\nu}\Lambda_\nu \,. \tag{34}$$

This can be written as a generalized gauge parameter transformation, characterized by the action of antisymmetric $\theta^{\mu\nu}$ bi-vector

$$\mathcal{O}_\theta\Lambda = \begin{pmatrix} \delta_\nu^\mu & \kappa\theta^{\mu\nu} \\ 0 & \delta_\mu^\nu \end{pmatrix} \begin{pmatrix} \xi^\nu \\ \Lambda_\nu \end{pmatrix} = \begin{pmatrix} \xi_\mu^R \\ \Lambda_\mu \end{pmatrix} . \tag{35}$$

The symmetry generator can be rewritten with new parameters in a new basis

$$\mathcal{G}_\mathcal{R}(\xi_R \oplus \Lambda) = \int d\sigma \Big[\xi_R^\mu \pi_\mu + \Lambda_\mu k^\mu\Big] . \tag{36}$$

It is obvious that the generator is the charge corresponding to current J_R (13) and that the T-duality relation $\mathcal{G}_\mathcal{R}(\xi_R \oplus \Lambda) \cong \mathcal{G}_\mathcal{C}(\xi \oplus \Lambda^C)$ holds.

There is a presence of non-geometric fluxes in basis algebra

$$\{k^\mu(\sigma), k^\nu(\bar\sigma)\} = -\kappa Q_\rho{}^{\mu\nu} k^\rho \delta(\sigma-\bar\sigma) - \kappa^2 R^{\mu\nu\rho}\pi_\rho\delta(\sigma-\bar\sigma) \,, \tag{37}$$

where $Q_\rho{}^{\mu\nu} = \partial_\rho\theta^{\mu\nu}$ is Q-flux and $R^{\mu\nu\rho} = \theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu}$ is R-flux. The new generator algebra [4, 10] is given by

$$\{\mathcal{G}_\mathcal{R}(\xi_1^R \oplus \Lambda_1), \mathcal{G}_\mathcal{R}(\xi_2^R \oplus \Lambda_2)\} = -\mathcal{G}_\mathcal{R}([\xi_1^R \oplus \Lambda_1, \xi_2^R \oplus \Lambda_2]_R) \,, \tag{38}$$

where the resulting bracket is the Roytenberg bracket [11]. It is a generalization of Courant bracket, obtained by twisting the Courant bracket by some bi-vector. The corresponding bi-vector in our case is the non-commutativity parameter $\kappa\theta^{\mu\nu}$. The Roytenberg bracket differs from the Courant bracket by

$$[\mathcal{O}_\theta(\xi_1 \oplus \Lambda_1), \mathcal{O}_\theta(\xi_2 \oplus \Lambda_2)]_C - \mathcal{O}_\theta[\xi_1 \oplus \Lambda_1, \xi_2 \oplus \Lambda_2]_C \,, \tag{39}$$

In its most general form, Roytenberg bracket encompasses all fluxes. In this case, canonical momenta are commutative, meaning that there is no H-flux present in basis algebra. The full expression is

$$\begin{aligned}
[\xi_1 \oplus \Lambda_1^R, \xi_2 \oplus \Lambda_2^R]_R = \Big(& [\xi_1,\xi_2]_L - \kappa[\xi_2, \Lambda_1^R\theta]_L + \kappa[\xi_1, \Lambda_2^R\theta]_L \\
& + \frac{\kappa^2}{2}[\theta,\theta]_S(\Lambda_1^R, \Lambda_2^R, .) - (\mathcal{L}_{\xi_2}\Lambda_1^R - \mathcal{L}_{\xi_1}\Lambda_2^R + \frac{1}{2}d(i_{\xi_1}\Lambda_2^R - i_{\xi_2}\Lambda_1^R))\kappa\theta\Big) \\
& \oplus \Big(\mathcal{L}_{\xi_1}\Lambda_2^R - \mathcal{L}_{\xi_2}\Lambda_1^R - \frac{1}{2}d(i_{\xi_1}b - i_{\xi_2}\Lambda_1^R) - [\Lambda_1^R, \Lambda_2^R]_\theta\Big) .
\end{aligned} \tag{40}$$

The expression $[\theta,\theta]_S(\Lambda_1^R, \Lambda_2^R, .)$ is the Schouten-Nijenhuis bracket [12] contracted with two 1-forms. It is a generalization of the Lie bracket on the space of multi-vectors. The expression $[\Lambda_1^R, \Lambda_2^R]_\theta$ is the Koszul bracket [13], a generalization of the Lie bracket on the space of differential forms.

We note that 2-form $2B_{\mu\nu}$ and bi-vector $\kappa\theta^{\mu\nu}$ used for twisting the Courant bracket in two cases are mutually T-dual (10). Moreover, both of these brackets appeared in considering charges corresponding to the generalized currents, defined in mutually T-dual bases. Therefore, we can conclude that the T-duality connects twisted Courant and Roytenberg bracket, as long as they are twisted by the mutually T-dual fields.

3. Conclusions

We considered a standard σ-model for closed bosonic string. We wrote Hamiltonian in terms of two types of currents. The components of these currents have been used as bases in which generalized currents were defined. These generalized currents are defined in mutually T-dual bases.

Afterwards, we considered the self T-dual symmetry generator $\mathcal{G}(\xi \oplus \Lambda)$. It takes the generalized gauge parameter $\xi \oplus \Lambda$ as a parameter and generates both general coordinate transformations and local gauge transformations. The Poisson bracket algebra of these generators was calculated and the Courant bracket has been obtained. The Courant bracket appeared in a same way in which Lie bracket was obtained in the case of general coordinate transformations algebra.

Next, we considered the action of B-transformation to the generalized gauge parameter. The symmetry generator written in terms of the resulting gauge parameter is the charge corresponding to the generalized current J_C. The Poisson bracket algebra of these generators gives rise to the Courant bracket twisted by a 2-form $2B$.

In the end, we consider the action of θ-transformation on the generalized gauge parameter. The newly obtained generator is the charge for the generalized current J_R and it gives rise to the Roytenberg bracket. The Roytenberg bracket in general represent the Courant bracket twisted by a bi-vector. In this specific case, the bi-vector is $\kappa\theta^{\mu\nu}$, which is T-dual to the Kalb-Ramond field $B_{\mu\nu}$. Consequently, we conclude that both the twisted Courant bracket and the Roytenberg bracket appear when generalized currents defined in two mutually T-dual bases are considered.

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