Nonassociative differential geometry and gravity*

Marija Dimitrijević Ćirić†
University of Belgrade, Faculty of Physics
Studentski trg 12-16, Belgrade, Serbia

Abstract

In this short contribution we introduce a nonassocaitive deformation of differential geometry and General Relativity. The nonassociativity is based on the string theory nongeometric R-flux. We use the twist formalism to consistently deform the algebra of infinitesimal diffeomorphisms into the quasi Hopf algebra of (deformed) infnitesimal diffeomorphisms and introduce the NA deformation of differential geometry. In particualr, we define the Levi-Civita connection, curvature tensor and torsion. The space-time quantities (curvature, torsion) are obtained by the zero momenum leaf projection to the space-time. The vacuum Einstein equation in space-time, expanded up to first order in the deformation parameter $\kappa\hbar$ is obtained.

1. Introduction

In the context of string theory, it is expected that the closed string sector provides a framework for a quantum theory of gravity. Namely, the massless bosonic modes of the closed string sector contain gravitational degrees of freedom such as the metric, the B-field, and the dilaton. In particular, in locally non-geometric backgrounds one expects to find a low-energy limit of closed string theory which is described by an effective nonassociative theory of gravity on spacetime.

Attempts to formulate a consistent effective gravity theory in the spacetime, starting from the nonasocaitive phase space of closed strings were done in [1, 2, 3]. There the construction is done using the twist approach. The twist approach provides a well defined way to introduce the noncommutative/nonassocaitive differential geometry and the notions of connections and curvature. The essential step [2, 3] is the projection from the phase space to the spacetime via the zero momentum leaf. In this short contribution we explain how the metric aspects of nonassociative differential

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† e-mail address: dmarija@ipb.ac.rs

geometry and vacuum Einstein equations in a model based on a locally non-geometric flux R are developed. The contribution is based on work done in [2, 3].

After a short overview of nonassociativity and noncommutativity in physics, in next section we develop the nonassocaitive differential geometry based on a cochain twist. This we use in Section 3 to construct a nonassociative theory of gravity on spacetime. Explicit expressions for the torsion, curvature, Ricci tensor and Levi-Civita connection in nonassociative Riemannian geometry on phase space are obtained. Using the projection to the zero momentum leaf, we construct the R-flux corrections to the Ricci tensor on spacetime, and comment on the potential implications of these corrections.

2. Nonassociative differential geometry

First ideas of a space-time with noncommuting (NC) coordinates appeared in the early days of quantum mechanics. In the 1930 Heisenberg proposed to introduce noncommuting coordinte operators in order to regularize the divergent electron self-energy. In the 1947 Snyder construced the first model of a NC space-time [4]. The renormalization theory, developed in the 1950es sucesfully solved the problem of divergences in quantum filed theory, so the idea of noncommutativity was not developed furhter. However, in the 1990es new results from string theory and searches for quantum gravity and quantum space-time renewed interest in noncommutative geometry. Lot of work has been done in formulating quantum field theory and gravity in a NC space-time, see reviews [5].

In the similar way, first ideas on nonassociativity of coordinates in physics go back to the early days of quantum mechanics. Jordan formulated a version of quantum mechanics [6] whith a new composition between herimitean observables

$$A \circ B = \frac{1}{2}((A+B)^2 - A^2 - B^2).$$

This composition gives a hermitean observable again, but it is nonassocaitive. Nambu further developed a modification of classical mechanics by introducing a Nambu-Poisson bracket $\{f, g, h\}$ instead of the usual Poisson bracket [7]. Nambu-Poisson bracket fulfils the fundamental identity instead of the usual Jacobi identity. The idea of nonassociativity in physics was then forgoten until recenty. Namley, it was discovered that symmetries of closed string field theory close a strong homotopy Lie-algebra, L_∞ algebra. These algebras can be seen as a generalization of the usual Lie algebra that do not fulfil the Jacobi identity, but higher homotopy relations instead. Nonassociative $\star$-products were develeoped from (closed) string theory in locally non-geometric backgrounds [8]. We will work with a model of a nonasocaitive phase space that originates from closed strings moving in a non-geometric background defined by the constant R-flux.

The phase space $\mathcal{M}$ has coordinates $x^A = (x^\mu, \tilde{x}_\mu = p_\mu)$, $\partial_A = \big(\partial_\mu, \tilde{\partial}^\mu = \frac{\partial}{\partial p_\mu}\big)$ with $A = 1, \ldots 2d$. We introduce a deformation by a cochain twist $\mathcal{F}$

$$\mathcal{F} = \exp\Big(-\tfrac{\mathrm{i}\,\hbar}{2}\,(\partial_\mu \otimes \tilde{\partial}^\mu - \tilde{\partial}^\mu \otimes \partial_\mu) - \tfrac{\mathrm{i}\,\kappa}{2}\, R^{\mu\nu\rho}\,(p_\nu\,\partial_\rho \otimes \partial_\mu - \partial_\mu \otimes p_\nu\,\partial_\rho)\Big)\ , \quad (1)$$

with $R^{\mu\nu\rho}$ totally antisymmetric and constant, and $\kappa := \frac{\ell_s^3}{6\hbar}$. This twist fails to fulfill the 2-cocycle condition

$$\Phi\,(\mathcal{F} \otimes 1)\,(\Delta \otimes \mathrm{id})\mathcal{F} = (1 \otimes \mathcal{F})\,(\mathrm{id} \otimes \Delta)\mathcal{F}\ . \quad (2)$$

The associator Φ is given by

$$\Phi = \exp\big(\hbar\kappa\, R^{\mu\nu\rho}\,\partial_\mu \otimes \partial_\nu \otimes \partial_\rho\big) =: \phi_1 \otimes \phi_2 \otimes \phi_3 = 1 \otimes 1 \otimes 1 + O(\hbar\kappa). \quad (3)$$

In the following text we will use the notation: $\mathcal{F} = \mathrm{f}^\alpha \otimes \mathrm{f}_\alpha$, $\mathcal{F}^{-1} = \bar{\mathrm{f}}^\alpha \otimes \bar{\mathrm{f}}_\alpha$, $\Phi^{-1} =: \bar{\phi}_1 \otimes \bar{\phi}_2 \otimes \bar{\phi}_3$. The R-matrix encodes the braiding and it is defined by

$$\mathcal{R} = \mathcal{F}^{-2} =: \mathrm{R}^{\,\alpha} \otimes \mathrm{R}_{\,\alpha}, \quad (4)$$

The inverse of the R-matrix is then

$$\mathcal{R}^{-1} = \mathcal{F}^2 =: \overline{\mathrm{R}}^{\,\alpha} \otimes \overline{\mathrm{R}}_{\,\alpha}.$$

The phase space $\mathcal{M}$ is invariant under the action of infinitesimal diffeomorphism. The Hopf algebra of infinitesimal diffeomorphisms $U\mathsf{Vec}(\mathcal{M})$ is given by:

$$\begin{aligned}
&[u, v] = (u^B \partial_B v^A - v^B \partial_B u^A)\partial_A,\\
&\Delta(u) = 1 \otimes u + u \otimes 1,\\
&\epsilon(u) = 0, S(u) = -u.
\end{aligned}$$

The twist (1) deforms this Hopf algebra into a quasi-Hopf algebra of infinitesimal diffeomorphisms $U\mathsf{Vec}^{\mathcal{F}}(\mathcal{M})$. The deformation is such that the algebra structure does not change, the coproduct is deformed

$$\Delta^{\mathcal{F}}\xi = \mathcal{F}\,\Delta\,\mathcal{F}^{-1},$$

while the counit and the antipod do not change: $\epsilon^{\mathcal{F}} = \epsilon, S^{\mathcal{F}} = S$.

On the basis vector fields the twist acts as

$$\begin{aligned}
\Delta_{\mathcal{F}}(\partial_\mu) &= 1 \otimes \partial_\mu + \partial_\mu \otimes 1\ ,\\
\Delta_{\mathcal{F}}(\tilde{\partial}^\mu) &= 1 \otimes \tilde{\partial}^\mu + \tilde{\partial}^\mu \otimes 1 + \mathrm{i}\,\kappa\, R^{\mu\nu\rho}\,\partial_\nu \otimes \partial_\rho\ .
\end{aligned}$$

We use the covariance under the infinitesimal diffeomorphisms as a guiding principle to define a NA phase space. Namley, we know that the differential geometry on $\mathcal{M}$ is covatiant under $U\mathsf{Vec}(\mathcal{M})$. Then we demand that the NA differential geometry on $\mathcal{M}$ should be covariant under the action of twist-deformed insintesimal diffeomorphism $U\mathsf{Vec}^{\mathcal{F}}(\mathcal{M})$. This means that for any $U\mathsf{Vec}(\mathcal{M})$-module algebra $\mathcal{A}$ (functions, forms, tensors) and for $a, b \in \mathcal{A}$, $u \in \mathsf{Vec}(\mathcal{M})$

$$u(ab) = u(a)b + au(b),$$

where the action on product is defined by the Leibniz rule, that is the coproducts of vectors u. The twist (1) deforms this into: $U\mathsf{Vec}(\mathcal{M}) \to U\mathsf{Vec}^{\mathcal{F}}(\mathcal{M})$ and $\mathcal{A} \to \mathcal{A}_\star$ with

$$ab \to a \star b = \bar{\mathrm{f}}^{\,\alpha}(a) \cdot \bar{\mathrm{f}}_{\,\alpha}(b).$$

Then $\mathcal{A}_\star$ is a $U\mathsf{Vec}^{\mathcal{F}}(\mathcal{M})$-module algebra

$$\xi(a \star b) = \xi_{(1)}(a) \star \xi_{(2)}(b),$$

for $\xi \in U\mathsf{Vec}^{\mathcal{F}}(\mathcal{M})$ and the action is via the twisted coproduct $\Delta^{\mathcal{F}}\xi = \xi_{(1)} \otimes \xi_{(2)}$.

The new composition is noncommutative

$$a \star b = \bar{\mathrm{f}}^{\,\alpha}(a) \cdot \bar{\mathrm{f}}_{\,\alpha}(b) = \bar{\mathrm{R}}^{\,\alpha}(b) \star \bar{\mathrm{R}}_{\,\alpha}(a) =: {}^{\alpha}b \star {}_{\alpha}a$$

and it is also nonassociative:

$$(a \star b) \star c = {}^{\phi_1}a \star ({}^{\phi_2}b \star {}^{\phi_3}c).$$

The noncommutativity is controled by the inverse of the R-matrix, while the nonassociativity is controled by the associator Φ.

In particular, the algebra of functions $C^\infty(\mathcal{M})$ is deformed to $C^\infty(\mathcal{M})_\star$

$$\begin{aligned} f \star g &= \bar{\mathrm{f}}^{\,\alpha}(f) \cdot \bar{\mathrm{f}}_{\,\alpha}(g) \qquad (5)\\ &= f \cdot g + \tfrac{\mathrm{i}\hbar}{2}\left(\partial_\mu f \cdot \tilde{\partial}^\mu g - \tilde{\partial}^\mu f \cdot \partial_\mu g\right) + \mathrm{i}\,\kappa\, R^{\mu\nu\rho}\, p_\nu\, \partial_\rho f \cdot \partial_\mu g + \cdots , \end{aligned}$$

For the special case of phase space coordinates we get

$$\begin{aligned} [x^\mu \overset{\star}{,} x^\nu] &= 2\,\mathrm{i}\,\kappa\, R^{\mu\nu\rho}\, p_\rho, \\ [p_\mu \overset{\star}{,} p_\nu] &= 0, \quad [x^\mu \overset{\star}{,} x^\nu] = \mathrm{i}\,\hbar\,\delta^\mu{}_\nu, \\ [x^\mu \overset{\star}{,} x^\nu \overset{\star}{,} x^\rho] &= \ell_s^3\, R^{\mu\nu\rho}. \end{aligned} \qquad (6)$$

The exterior algebra of differential forms $\Omega^\sharp(\mathcal{M})$ is deformed to $\Omega^\sharp(\mathcal{M})_\star$ with

$$\begin{aligned} \omega \wedge_\star \eta &= \bar{\mathrm{f}}^{\,\alpha}(\omega) \wedge \bar{\mathrm{f}}_{\,\alpha}(\eta), \qquad (7)\\ f \star \mathrm{d}x^A &= \mathrm{d}x^C \star \left(\delta^A{}_C\, f - \mathrm{i}\,\kappa\, \mathscr{R}^{AB}{}_C\, \partial_B f\right) , \end{aligned}$$

with non-vanishing components $\mathscr{R}^{x^\mu, x^\nu}{}_{\tilde{x}_\rho} = R^{\mu\nu\rho}$. Especcially, for the basis 1-forms

$$\begin{aligned} &(\mathrm{d}x^A \wedge_\star \mathrm{d}x^B) \wedge_\star \mathrm{d}x^C = {}^{\phi_1}(\mathrm{d}x^A) \wedge_\star \left({}^{\phi_2}(\mathrm{d}x^B) \wedge_\star {}^{\phi_3}(\mathrm{d}x^C)\right) \\ &= \mathrm{d}x^A \wedge_\star (\mathrm{d}x^B \wedge_\star \mathrm{d}x^C) = \mathrm{d}x^A \wedge \mathrm{d}x^B \wedge \mathrm{d}x^C. \end{aligned}$$

Exterior derivative d is undeformed. It fulfills $\mathrm{d}^2 = 0$ and the undeformed Leibniz rule

$$\mathrm{d}(\omega \wedge_\star \eta) = \mathrm{d}\omega \wedge_\star \eta + (-1)^{|\omega|}\, \omega \wedge_\star \mathrm{d}\eta. \tag{8}$$

Duality or the $\star$-pairing is defined as

$$\langle\, \omega\, ,\, u\, \rangle_\star = \left\langle\, \overline{\mathrm{f}}^{\,\alpha}(\omega)\, ,\, \overline{\mathrm{f}}_{\,\alpha}(u)\, \right\rangle\, . \tag{9}$$

Finally, the Lie-derivative can consistently be deformed to a $\star$-Lie drivative as

$$\begin{aligned}
&\mathcal{L}^\star_u(T) = \mathcal{L}_{\overline{\mathrm{f}}^{\,\alpha}(u)}(\overline{\mathrm{f}}_{\,\alpha}(T)), \qquad (10)\\
&\mathcal{L}^\star_u(\omega \wedge_\star \eta) = \mathcal{L}^\star_{\bar\phi_1 u}(^{\bar\phi_2}\omega) \wedge_\star {}^{\bar\phi_3}\eta + {}^{\alpha}(^{\bar\phi_1\,\bar\varphi_1}\omega) \wedge_\star \mathcal{L}^\star_{\alpha(^{\bar\phi_2\,\bar\varphi_2}u)}(^{\bar\phi_3\,\bar\varphi_3}\eta),\\
&[\mathcal{L}^\star_u, \mathcal{L}^\star_v]_\bullet = [\overline{\mathrm{f}}^{\,\alpha}\mathcal{L}^\star_u, \overline{\mathrm{f}}_{\,\alpha}\mathcal{L}^\star_v] = \mathcal{L}^\star_{[u,v]_\star},
\end{aligned}$$

with $[u,v]_\star = \left[\overline{\mathrm{f}}^{\,\alpha}(u), \overline{\mathrm{f}}_{\,\alpha}(v)\right]$ and

$$\big[u, [v,z]_\star\big]_\star = \big[[^{\bar\phi_1}u, {}^{\bar\phi_2}v]_\star, {}^{\bar\phi_3}z\big]_\star + \big[^{\alpha}(^{\bar\phi_1\,\bar\varphi_1}v), [_{\alpha}(^{\bar\phi_2\,\bar\varphi_2}u), {}^{\bar\phi_3\,\bar\varphi_3}z]_\star\big]_\star.$$

It is well known that the usual Lie derivative $\mathcal{L}_u$ generates a one parameter family of diffeomorphisms. However, a relation of $\mathcal{L}^\star_u$ with diffeomorphism symmetry in space-time still needs to be understood fully [9, 10].

After formulating the basic notions of the NA differential geometry, we now define a $\star$-connection by

$$\begin{aligned}
\nabla^\star\, :\, \mathsf{Vec}_\star \quad &\longrightarrow \quad \mathsf{Vec}_\star \otimes_\star \Omega^1_\star\\
u \quad &\longmapsto \quad \nabla^\star u\ , \qquad (11)\\
\nabla^\star(u \star f) \quad &= \quad \big(^{\bar\phi_1}\nabla^\star(^{\bar\phi_2}u)\big) \star {}^{\bar\phi_3}f + u \otimes_\star \mathrm{d}f. \qquad (12)
\end{aligned}$$

A connection defined in this way satisfies the right Leibniz rule, for $u \in \mathsf{Vec}_\star$ and $f \in A_\star$. In particular

$$\begin{aligned}
&\nabla^\star\partial_A =: \partial_B \otimes_\star \Gamma^B_A =: \partial_B \otimes_\star (\Gamma^B_{AC} \star \mathrm{d}x^C)\ . \qquad (13)\\
&\mathrm{d}_{\nabla^\star}(\partial_A \otimes_\star \omega^A) = \partial_A \otimes_\star (\mathrm{d}\omega^A + \Gamma^A_B \wedge_\star \omega^B),
\end{aligned}$$

for $\omega^A \in \Omega^\sharp_\star$.

Once we defined the $\star$-connection, the torsion and the curvature tensors can be defined straightforwardly. The torsion we define as

$$\begin{aligned}
&\mathsf{T}^\star := \mathrm{d}_{\nabla^\star}\big(\partial_A \otimes_\star \mathrm{d}x^A\big)\, :\, \mathsf{Vec}_\star \otimes_\star \mathsf{Vec}_\star \to \mathsf{Vec}_\star,\\
&\mathsf{T}^\star(\partial_A, \partial_B) = \partial_C \star (\Gamma^C_{AB} - \Gamma^C_{BA}) =: \partial_C \star \mathsf{T}^C{}_{AB}.
\end{aligned}$$

In the coordinate basis, the torsion-free condition reduces to $\Gamma^C_{AB} = \Gamma^C_{BA}$.

The curvature tensor we define as

$$\begin{aligned}
&\mathsf{R}^\star := \mathrm{d}_{\nabla^\star} \bullet \mathrm{d}_{\nabla^\star}\, :\, \mathsf{Vec}_\star \ \longrightarrow\ \mathsf{Vec}_\star \otimes_\star \Omega^2_\star,\\
&\mathsf{R}^\star(\partial_A) = \partial_C \otimes_\star (\mathrm{d}\Gamma^C_A + \Gamma^C_B \wedge_\star \Gamma^B_A) = \partial_C \otimes_\star \mathsf{R}^C_A,
\end{aligned}$$

The Ricci tensor is a contraction of the curvature tensor

$$\begin{aligned}\mathsf{Ric}^\star(u,v) &:= -\langle\, \mathsf{R}^\star(u,v,\partial_A)\,,\, \mathrm{d}x^A\,\rangle_\star \\ \mathsf{Ric}^\star &= \mathsf{Ric}_{AD}\star(\mathrm{d}x^D\otimes_\star \mathrm{d}x^A).\end{aligned} \tag{14}$$

Commponents of the Ricci tensor in the coordinate basis can be calculated from $\mathsf{Ric}_{BC} := \mathsf{Ric}^\star(\partial_B,\partial_C)$ and they are given by

$$\begin{aligned}\mathsf{Ric}_{BC} &= \partial_A\Gamma^A_{BC} - \partial_C\Gamma^A_{BA} + \Gamma^A_{B'A}\star\Gamma^{B'}_{BC} - \Gamma^A_{B'C}\star\Gamma^{B'}_{BA} \\ &+ \mathrm{i}\,\kappa\,\Gamma^A_{B'E}\star\big(\mathscr{R}^{EG}{}_A\,(\partial_G\Gamma^{B'}_{BC}) - \mathscr{R}^{EG}{}_C\,(\partial_G\Gamma^{B'}_{BA})\big) \\ &+ \mathrm{i}\,\kappa\,\mathscr{R}^{EG}{}_A\,\partial_G\partial_C\Gamma^A_{BE} - \mathrm{i}\,\kappa\,\mathscr{R}^{EG}{}_A\,\partial_G\big(\Gamma^A_{B'E}\star\Gamma^{B'}_{BC} - \Gamma^A_{B'C}\star\Gamma^{B'}_{BE}\big) \\ &+ \kappa^2\,\mathscr{R}^{AF}{}_D\,\big(\mathscr{R}^{EG}{}_A\,\partial_F(\Gamma^D_{B'E}\star\partial_G\Gamma^{B'}_{BC}) - \mathscr{R}^{EG}{}_C\,\partial_F(\Gamma^D_{B'E}\star\partial_G\Gamma^{B'}_{BA})\big)\ .\end{aligned} \tag{15}$$

Unfortunately, a scalar curvature cannot be defined along these lines. That is, it cannot be seen as a map and the inverse metric tensor needed for the definiton. Due to nonassociativity it is not straightforward to define the inverse of the metric tensor. Namely

$$G^{MN}\star G_{NP} = \delta^P_M, \;\; \text{but} \;\; (G^{MN}\star G_{NP})\star f \neq G^{MN}\star(G_{NP}\star f).$$

This problem we hope to solve in our future work.

3. Nonassociative deformation of General Relativity

The connection of General Relativity $\Gamma^{\mathsf{LC}\,\rho}_{\mu\nu}$ is a Levi-Civita connection: it is torssion-free and metric compatible $\nabla_\alpha g_{\mu\nu} = 0$. In this section we generalize this conditions to the NA phase space. First we define a metric tensor $\mathsf{g}^\star \in \Omega^1_\star \otimes_\star \Omega^1_\star$ and then we demand

$$^\star\nabla\mathsf{g}^\star = 0. \tag{16}$$

In addition, the connection is torsion free $\Gamma^C_{AB} = \Gamma^C_{BA}$. Expanding the above condition up to first order in $\hbar\kappa$ we find

$$\begin{aligned}\Gamma^{S(0,0)}_{AD} &= \Gamma^{\mathsf{LC}\,S}_{AD} = \tfrac{1}{2}\,\mathsf{g}^{SQ}\,(\partial_D\mathsf{g}_{AQ} + \partial_A\mathsf{g}_{DQ} - \partial_Q\mathsf{g}_{AD})\ , \\ \Gamma^{S(0,1)}_{AD} &= -\frac{\mathrm{i}\,\hbar}{2}\,\mathsf{g}^{SP}\,\big((\partial_\mu\mathsf{g}_{PQ})\,\tilde\partial^\mu\Gamma^{\mathsf{LC}\,Q}_{AD} - (\tilde\partial^\mu\mathsf{g}_{PQ})\,\partial_\mu\Gamma^{\mathsf{LC}\,Q}_{AD}\big)\ , \\ \Gamma^{S(1,0)}_{AD} &= \mathrm{i}\,\kappa\,R^{\alpha\beta\gamma}\left(\tilde{\mathsf{g}}^S{}_\gamma\,\mathsf{g}_{\beta N}\,\big(\partial_\alpha\Gamma^{\mathsf{LC}\,N}_{AD}\big) - \mathsf{g}^{SM}\,p_\beta\,(\partial_\gamma\mathsf{g}_{MN})\,\partial_\alpha\Gamma^{\mathsf{LC}\,N}_{AD}\right)\ , \\ \Gamma^{S(1,1)}_{AD} &= \frac{\hbar\,\kappa}{2}\,R^{\alpha\beta\gamma}\Big[\ \text{long expression}\ + (\partial_\alpha\mathsf{g}^{SQ})\,(\partial_\beta\mathsf{g}_{QP})\,\partial_\gamma\Gamma^{\mathsf{LC}\,P}_{AD}\Big].\end{aligned} \tag{17}$$

Here we labeled $\tilde{\mathsf{g}}^S{}_\gamma = \mathsf{g}^{SM}\,\delta_{M,\tilde x_\gamma}$. Notice that $\Gamma^{S(0,1)}_{AD}$ and $\Gamma^{S(1,0)}_{AD}$ are purely imaginary, while $\Gamma^{S(1,1)}_{AD}$ is real. For g_{MN} that does not depend on the momenta p_μ, only the last term in $\Gamma^{S(1,1)}_{AD}$ remains.

The phase space vacuum Einstein equations are given by

$$\mathsf{Ric}_{BC} = 0\ . \tag{18}$$

Using (14) and (17) we obtain vacuum Einstein equations in phase space expanded up to first order in $\hbar\kappa$. However, we are interested in the deformation of General Relativity in space-time. To extract the space-time vacuum Einstein equations from (18) we develop a method of projection which we describe in the following.

We start from objects in the space-time M, for example the space-time metric tensor $g = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$. Then we lift these objects to the phase space $\mathcal{M}$ foliated with leaves of constant momenta. Note that each leave is diffeomorphic to M. We do all calculation using the NA differential geometry tools developed in the previous section. Final results (in phase space) we project to the space-time, using again a leaf of constant momenta. In particular, we will use the zero-momentum leaf to perform the projection. Our procedure is ilustrated by the diagram

$$\begin{tikzcd}
C^\infty(\mathcal{M}) \arrow[r, "Q"] & \widehat{C^\infty(\mathcal{M})} \arrow[d, "s^*_{\bar p}=\sigma^*"] \\
C^\infty(M) \arrow[u, "\pi^*"] \arrow[r, dotted, "Q_{\bar p}"'] & \widehat{C^\infty(M)}
\end{tikzcd}$$

Applying this procedure to the metric tensor $g = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ gives the phase space metric tensor $\hat{\mathsf{g}}_{MN}\,\mathrm{d}x^M \otimes \mathrm{d}x^N$ with

$$\big(\hat{\mathsf{g}}_{MN}(x)\big) = \begin{pmatrix} \mathsf{g}_{\mu\nu}(x) & 0 \\ 0 & \mathsf{h}^{\mu\nu}(x) \end{pmatrix}\ . \tag{19}$$

To have a non-singular metric we had to introduce an additional nondegenerate bilinear $\mathsf{h}(x)^{\mu\nu}\,\mathrm{d}\tilde{x}_\mu \otimes \mathrm{d}\tilde{x}_\nu$. Its choice is arbitrary, we pick up the simplest and the most natural choice $\mathsf{h}(x)^{\mu\nu} = \eta^{\mu\nu}$.

Now we can do all calculations in phase space, using the NA differential geometry. In particular, we calculate Ric_{BC} in terms of g_{AB}, (14), (17). In the end we project the result to space-time using the zero section $x \mapsto \sigma(x) = (x, 0)$. In particular, the projection of Ricci tensoris given by

$$\begin{aligned} \mathsf{Ric} \quad &\to \quad \mathsf{Ric}^{\star\circ} = \mathsf{Ric}^\circ_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu, \\ \mathsf{Ric}^\circ_{\mu\nu}(x) \quad &= \quad \sigma^*(\mathsf{Ric}_{\mu\nu})(x,p) = \mathsf{Ric}_{\mu\nu}(x,0). \end{aligned}$$

Components of the lifted metric $\hat{\mathsf{g}}_{MN}\,\mathrm{d}x^M \otimes \mathrm{d}x^N = \mathsf{g}_{MN}\star(\mathrm{d}x^M \otimes_\star \mathrm{d}x^N)$, expanded up to first order in $\hbar\kappa$ are given by

$$\mathsf{g}_{MN}(x) = \begin{pmatrix} \mathsf{g}_{\mu\nu}(x) & \frac{\mathrm{i}\,\kappa}{2}\,R^{\sigma\nu\alpha}\,\partial_\sigma \mathsf{g}_{\mu\alpha} \\ \frac{\mathrm{i}\,\kappa}{2}\,R^{\sigma\mu\alpha}\,\partial_\sigma \mathsf{g}_{\alpha\nu} & \eta^{\mu\nu}(x) \end{pmatrix}\ . \tag{20}$$

Finally, components of the Ricci tensor in space-time, expanded up to first order in $\hbar\kappa$ follow as:

$$\begin{aligned}
\mathsf{Ric}^{\circ}_{\mu\nu} = \mathsf{Ric}^{\mathsf{LC}}_{\mu\nu} + \tfrac{\ell_s^3}{12}\, R^{\alpha\beta\gamma} \Big(& \partial_\rho\big(\partial_\alpha \mathsf{g}^{\rho\sigma}\,(\partial_\beta \mathsf{g}_{\sigma\tau})\,\partial_\gamma \Gamma^{\mathsf{LC}\,\tau}_{\mu\nu}\big) \\
& -\partial_\nu\big(\partial_\alpha \mathsf{g}^{\rho\sigma}\,(\partial_\beta \mathsf{g}_{\sigma\tau})\,\partial_\gamma \Gamma^{\mathsf{LC}\,\tau}_{\mu\rho}\big) \\
+\,\partial_\gamma \mathsf{g}_{\tau\omega}\big(& \partial_\alpha(\mathsf{g}^{\sigma\tau}\,\Gamma^{\mathsf{LC}\,\rho}_{\sigma\nu})\,\partial_\beta \Gamma^{\mathsf{LC}\,\omega}_{\mu\rho} - \partial_\alpha(\mathsf{g}^{\sigma\tau}\,\Gamma^{\mathsf{LC}\,\rho}_{\sigma\rho})\,\partial_\beta \Gamma^{\mathsf{LC}\,\omega}_{\mu\nu} \\
+\,(\Gamma^{\mathsf{LC}\,\sigma}_{\mu\rho}\,&\partial_\alpha \mathsf{g}^{\rho\tau} - \partial_\alpha \Gamma^{\mathsf{LC}\,\sigma}_{\mu\rho}\,\mathsf{g}^{\rho\tau})\,\partial_\beta \Gamma^{\mathsf{LC}\,\omega}_{\sigma\nu} \\
-\,(\Gamma^{\mathsf{LC}\,\sigma}_{\mu\nu}\,&\partial_\alpha \mathsf{g}^{\rho\tau} - \partial_\alpha \Gamma^{\mathsf{LC}\,\sigma}_{\mu\nu}\,\mathsf{g}^{\rho\tau})\,\partial_\beta \Gamma^{\mathsf{LC}\,\omega}_{\sigma\rho}\big)\Big).
\end{aligned} \tag{21}$$

The vacuum Einstein equations in space-time are given by

$$\mathsf{Ric}^{\circ}_{\mu\nu} = 0. \tag{22}$$

4. Conclusions

In this short contribution, we described how the R-flux (via NA differential geometry) generates non-trivial dynamical consequences on spacetime. The first order corrections are are independent of $\hbar$ (they are first order in $\kappa\hbar = \frac{\ell_s^3}{6}$) and real-valued. To obtain the results in space-time, we used the projection to the zero momentum leaf. Note that pulling back to a leaf of constant momentum $p = p^\circ$ (generally) gives a non-vanishing imaginary contribution $\mathsf{Ric}^{(1,0)}_{\mu\nu}\big|_{p=p^\circ}$ to the spacetime Ricci tensor. Also, n-triproducts calculated on the zero momentum leaf [2] coincide with those proposed in [8].

We took the simplest choice of $\mathsf{h}(x)^{\mu\nu} = \eta^{\mu\nu}$. This can be changed to a more general metric. In relation with Born geometry discussed in [11] we can say that in our model nonassociativity does not generates curved momentum space. It might generate a more genral deformations with $\mathsf{h}(x)^{\mu\nu} \neq \eta^{\mu\nu}$. This has to be investigated further.

There are lots of projects to be discussed in our future work. Some of them are: phenomenological consequences of the R-flux induced corrections to GR solutions, construction of scalar curvature, adding matter fields, full Einstein equations. The twisted diffeomorphism symmetry has to be understood better, in particular its relation with the L_∞ structure.

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