Hyperspherical three-body variables applied to Sakumichi and Suganuma's lattice QCD data*

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Abstract

We analyse the 2015 lattice QCD calculations of the three-quark potential by Sakumichi and Suganuma [1] using hyperspherical three-body variables, following Refs. [2, 3]. We show that the triangle shape-dependence of Sakumichi & Suganuma's three-quark confinement potential evaluated at $\beta = 5.8$ differs from the one evaluated at $\beta = 6.0$ by 2%. This difference corresponds precisely to the difference between the Y- and Δ-string potentials in the isosceles and right-angled triangle geometries, thus setting an upper and a lower bound on the mean value.

1. Introduction

In spite of decades-long efforts, [1, 4, 5, 6], the functional form of the three-heavy-quark potential in lattice QCD is still not well known. Unlike

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the confining part of the two-body potential, which depends only on one variable - the distance between two bodies (quark and antiquark) - the three-body potential depends on three (scalar) variables.

These three variables may be defined in different ways, perhaps the simplest one being the set of three two-body separations. The hyperspherical coordinates, see [2, 3] and references therein, consist of a (dimensional) hyperradius $R = \sqrt{\frac{1}{3}\sum_{i<j}^{3}(\mathbf{r}_i - \mathbf{r}_j)^2} = \sqrt{\sum_{i}^{3}(\mathbf{R}_{\text{g.b.}} - \mathbf{r}_i)^2}$, which is proportional to the root-mean-square distance of the three particles from their geometrical barycenter $\mathbf{R}_{\text{g.b.}} = \frac{1}{3}\sum_{i}^{3}\mathbf{r}_i$ [1], and scales linearly with λ: $R \to \lambda R$ and thus measures the size of the system; and of two dimensionless (shape) variables, that may be expressed as hyperangles, or some functions thereof. The scaling transformation affects only the hyperradius, whereas the permutation symmetry affects only the two "shape space" variables/hyperangles.

Thus far, most of analyses of lattice 3-body potentials had been in terms of (only) one scalar variable, chosen according to the tastes and prejudices of the authors, usually as the Δ- or Y-string lengths, l_Δ, l_Y, respectively. As both l_Δ and l_Y are linearly proportional to the hyperradius R, both parametrizations effectively "measure" the linear hyperradial dependence of the confining 3-body potential, averaged over some set of shape-space points. Thus, agreement, or disagreement of such a fit with the lattice data must depend on the choice of shape-space points.

The Δ- and Y-string lengths, l_Δ, l_Y, differ little in most of the shape space, the relative difference being the largest near and at "two-body collision points", where the Coulomb interaction also peaks. Thus, an effective disambiguation between Δ- and Y-string potentials demands either a very precise lattice measurement in regions where the difference is small, or a very precise subtraction of the Coulomb term in regions where the difference is large. In either case, the task is difficult. Thus, the final results may depend on the initial choice of geometrical configurations on the lattice, in addition to the usual statistical and systematic errors/uncertainties. These geometrical configurations have been generally subject of entirely arbitrary choice(s) of various authors.

We have analysed Koma & Koma's and Takahashi et al.'s lattice results [4, 5] in terms of three hyperspherical coordinates for the first time, in Ref. [2, 3]. Whereas the old Takahashi et al. data was insufficient to produce a continuous functional shape-space dependence, Koma & Koma's [4] results yielded two continuous, generally smooth functional dependences along two orthogonal lines in the shape space. There are several important differences between Komas' and Sakumichi's calculations: [4] had a 24^4 lattice at $\beta = 6.0$, with 221 three-quark geometries and only one gauge configuration; [1] had a $16^3 \times 32$ lattice at $\beta = 5.8$ with 101 three-quark

[1] which equals the physical center-of-mass $\mathbf{R}_{\text{g.b.}} = \mathbf{R}_{\mathbf{CM}}$ when all three masses are equal

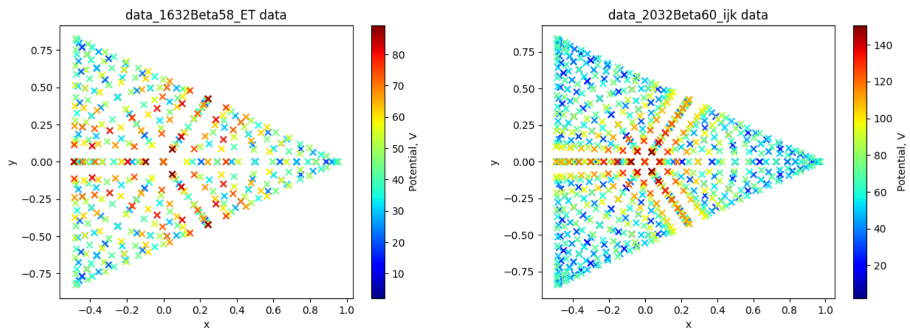

Figure 1: Distribution of Sakumichi & Suganuma's 3q configurations at $\beta = 5.8$ (l.h.s.) and $\beta = 6.0$ (r.h.s.).

geometries, and a $20^3 \times 32$ lattice at $\beta = 6.0$ and 211 three-quark geometries, with 1000 and 2000 gauge configurations, respectively.

2. Analysis of lattice data

In this report we subject Sakumichi & Suganuma's lattice data [1] to the same kind of analysis as Koma & Koma's [4] in Refs. [2, 3]. Fig. 1 depicts all configurations of the three-body system in Refs. [1, 4] including their permutations. The three lines that cross the origin represent isosceles triangles. Three lines orthogonal to them represent the right-angled triangles - one such line is at $y = -0.5$. One can see here that these lines are the only two sets of geometric configurations that are common to both Koma and Suganuma data. Therefore, we shall use them both.

We assume that the total three-quark potential V_{3q} has the form

$$V_{3q}(\alpha, \phi, R) = -\frac{A(\alpha, \phi)}{R} + B(\alpha, \phi)R + C, \tag{1}$$

henceforth referred to as the Coulomb + linear potential Ansatz. The first term represents the sum of QCD Coulomb pairwise interactions, which is dominant at small values of the hyper-radius R. The second term represents the confinement potential, which is linear in R and dominant at large values of hyper-radius R, and the third term - C - is a constant. Here $A(\phi, \alpha)$ is assumed to be the (standard) sum of of pair-wise Coulomb terms, and $B(\phi, \alpha)$ is the unknown hyper-angular dependence of the linearly rising confining potential. Our goal is to determine $B(\phi, \alpha) \simeq V/R$ using the lattice data and the well-known hyper-angular and hyper-radial dependences of the two-body Coulomb term, as explained in Refs. [2, 3].

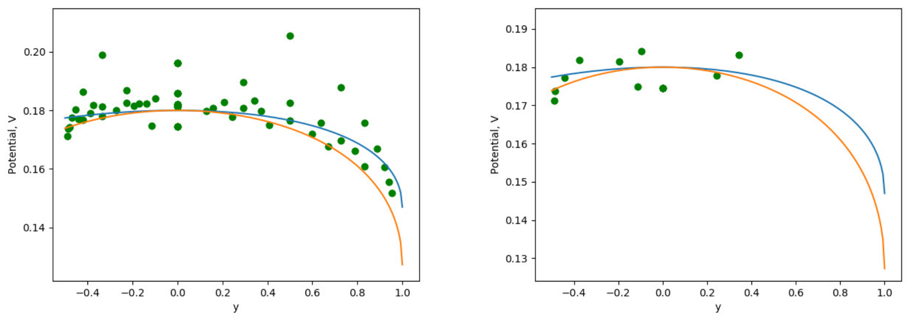

Figure 2: Extracted values of Sakumichi & Suganuma's V/R for isosceles triangles ($y = 0$), at $\beta = 5.8$: a) all (l.h.s); b) with hyper-radius filter $R \geq 8.0$ (r.h.s). The blue line represents the Δ-string, the yellow line is the Y-string.

2.1. The $\beta = 5.8$ case

It can be seen in Fig. 2.a that for the isosceles triangle configurations in the [1] $\beta = 5.8$ data set, all of the $B(x)$ values form a scattered set of points, some of which, though not all, lie between the Δ and Y-string potentials' functional forms. As one imposes the hyperradial filter $R \geq 8.0$, the scatter reduces, as does the number of points, Fig. 2.b, but no single, smooth curve emerges. The corresponding graphs for the right-angled triangles are shown in Fig. 3. Note that the hyperradial filter $R \geq 7.0$ eliminates many of the low-lying points and that almost all of the remaining points fall onto the (upper, blue) Δ-string prediction. The only discrepancy are the multiple points at the isosceles right-angled configuration.

2.2. The $\beta = 6.0$ case

Similarly: a) (again) the hyperradial filter $R \geq 8.0$ eliminates the many widely scattered low-R points in Fig. 4, and almost all of the remaining points fall onto the (lower, yellow) Y-string prediction in Fig. 5; b) the resulting y dependence of $V(y)/R$ for right-angled triangles in Fig. 4 is practically equivalent to the Y-string (the discrepancies are several multiple points near the isosceles right-angled configuration), and differs markedly from that at $\beta = 5.8$ in Fig. 2. This difference is (almost entirely) reducible to a constant shift.

3. Conclusions

We see that the results of our analysis are beset with several ambiguities:

1) Sakumichi & Suganuma [1] have calculated the potential at two different values of the coupling constant $\beta = 5.8$ and $\beta = 6.0$, but in two

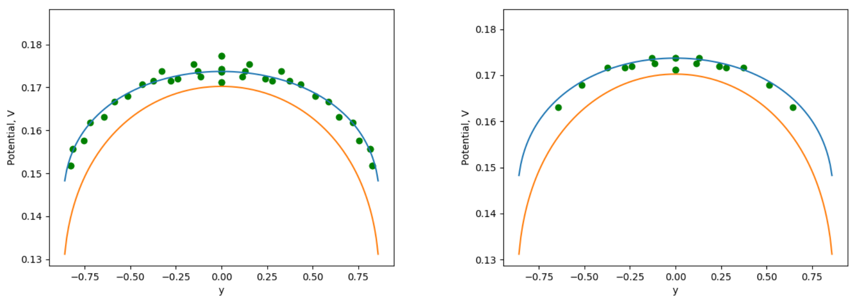

Figure 3: Extracted values of V/R for: a) all right triangles ($x = 0.5$) (l.h.s.) b) right triangles ($x = 0.5$) with hyper-radius filter $R \geq 7.0$, at $\beta = 5.8$, (r.h.s.). The blue line represents the Δ-string, yellow line is the Y-string.

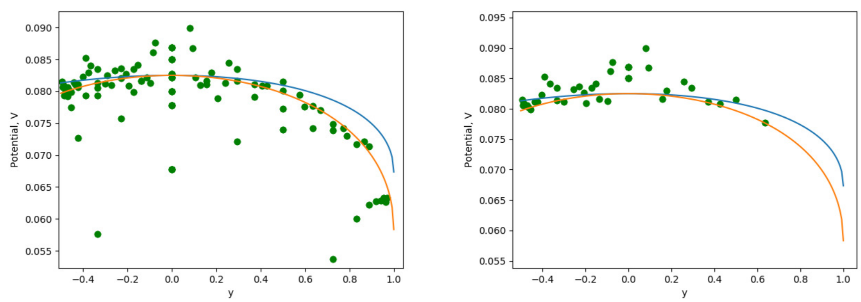

Figure 4: Extracted values of V/R for: isosceles triangles ($y = 0$), at $\beta = 6.0$. a) unrestricted data (l.h.s.); b) with hyper-radius filter $R \geq 9.0$. (r.h.s.).

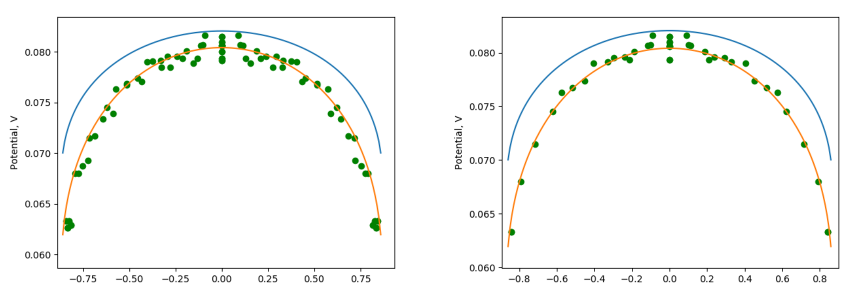

Figure 5: Extracted values of V/R for: a) all right triangles ($x = 0.5$) (l.h.s.) b) right triangles ($y = 0$) with hyper-radius larger than 8.0, at $\beta = 6.0$, (r.h.s.).

somewhat different sets of geometric and gauge configurations. These two calculations ought to agree with each other after rescaling (renormalization), which they do not (in their entirety), see Figs. 3, 5 and Figs. 2 and 4.

2) Sakumichi & Suganuma's [1] geometric configurations differ somewhat from Koma & Koma's [4], even in the small region of overlap. Namely, none of Sakumichi's right-angled triangles are exactly right-angled: there is always a small discrepancy, which leaves open the question of the correct normalization of the confining potential V/R in Figs. 3 and 5.

The shape-dependent part of the confining potential V/R must/can not change under scaling/renormalization, [7], therefore the $\beta = 5.8$ and $\beta = 6.0$ functional dependences must be identical, which they are not. The difference between the two potentials is consistent with an overall (constant) shift, however, which might be reconcilable with statistical and/or systematic errors.

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