From 3D torus with H-flux to torus with R-flux and back*

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Abstract

In this article we study 3D closed bosonic string propagating in the constant metric and Kalb-Ramond field with one non-zero component, $B_{xy} = Hz$, where field strength H is infinitesimal. We will T-dualize along line $x \to y \to z$, which means that we T-dualize first along x coordinate, then along y and, finally, along z coordinate. After first two T-dualizations we obtain Q flux theory which is just locally well defined, while after all three T-dualizations we obtain nonlocal R flux theory. The Q flux theory is commutative one and the R flux theory is noncommutative and nonassociative one. After that we reverse the T-dualization line and T-dualize along $z \to y \to x$. All three theories are nonlocal, but after the first T-dualization we obtain commutative and associative theory, while after we T-dualize along y, we get noncommutative and associative theory. T-dualizing along x, we come to the theory which is both noncommutative and nonassociative. The form of the final T-dual action does not depend on the order of T-dualization while noncommutativity and nonassociativity relations could be obtained from those in the $x \to y \to z$ case by replacing $H \to -H$.

1. Introduction

Heisenberg suggested coordinate noncommutativity in order to solve the problem of infinities before developing of renormalization procedure. In the paper [1] discrete Lorentz invariant space-time is constructed, which means that coordinates are noncommutative.

Noncommutativity came into the focus of interest with the appearance of the paper [2], where it is shown that open string endpoints in the presence

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of the constant metric and Kalb-Ramond field became noncommutative. After this article many others [3] appeared addressing the same subject but using different approaches.

The closed bosonic string in the presence of constant metric and Kalb-Ramond filed remains commutative, because there are no boundary conditions constraining string dynamics. Noncommutativity could be achieved [4] but using T-duality procedure and coordinate dependent Kalb-Ramond field.

T-duality as a fundamental feature of string theory [5, 6, 7, 8, 9, 10, 11] is realized within Buscher T-dualization procedure [6] which can considered as definition of T-dualization. It is applicable along directions on which background fields do not depend. In order to work with coordinate dependent backgrounds generalized T-dualization procedure is developed [12, 13, 14].

Here we will study closed bosonic string in the presence of the constant metric and linear dependent Kalb-Ramond field with just one nonzero component, $B_{xy} = Hz$, the background already analyzed in [15]. In all calculations we keep constant and linear terms in infinitesimal field strength H. We will use transformation laws, relations which connect initial and T-dual variables, in canonical form, expressed in terms of the coordinates and momenta. Our task is to T-dualize along T-dualization chain $x \to y \to z$ and in opposite direction and examine the influence of the T-dualization sequence on the form of the final theory (theory obtained after three T-dualizations) as well as on the noncommutativity and nonassociativity parameters.

T-dualizations along x and y produces the Q-flux background [15], which is still locally well defined, but the theory is commutative. Applying the generalized T-duality procedure [12, 13, 14], z T-dualization gives R flux theory which is nonlocal one because it depends on the non-locally defined variable ΔV. Nonzero Poisson brackets of the T-dual coordinates shows that there is a connection of non-locality and closed string noncommutativity.

The form of noncommutativity is proportional to the infinitesimal field strength H and difference of the initial coordinates. When arguments of the coordinates are different, $\sigma \neq \bar{\sigma}$, there exists noncommutativity. The consequence of the coordinate dependent noncommutativity relations is broken Jacobi identity - nonassociativity occurs. Nonassociativity parameter is proportional to the field strength H.

In the second part of the article we will T-dualize first along z and then along isometry directions y and finally along x. After first T-dualization we get commutative and associative theory as in xyz case. The second T-dualization produces noncommutative and associative theory. In the xyz case, theory second in the T-dualization chain is both commutative and associative. The action of the final theory is the same as in xyz case which is nonassociative and noncommutative. The noncommutativity and nonassociativity parameters have one additional "$-$" sign comparing with the corresponding ones in [16].

2. Action and T-dualization procedure

In this section we will present the construction of the model and give some important details of the T-dualization procedure.

2.1. Model

The closed bosonic string action is of the form [5]

$$S = \kappa \int_{\Sigma} d^2\xi \sqrt{-g} \left\{ \left[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}(x) + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}(x) \right] \partial_\alpha x^\mu \partial_\beta x^\nu + \Phi(x) R^{(2)} \right\} , \tag{1}$$

where world-sheet surface Σ is parameterized by $\xi^\alpha = (\tau , \sigma)$ $[(\alpha = 0 , 1)$, $\sigma \in (0 , \pi)]$, while x^μ $(\mu = 0, 1, 2, \ldots, D-1)$ are space-time coordinates. Intrinsic world sheet metric is denoted by $g_{\alpha\beta}$, and the corresponding scalar curvature with $R^{(2)}$. Here $G_{\mu\nu}$ is, in the general case, coordinate dependent metric, $B_{\mu\nu}$ is coordinate dependent Kalb-Ramond field, while Φ is dilaton field.

If we intend to have conformal symmetry on the quantum level, background fields are not arbitrarily chosen i.e. they must obey the space-time field equations [17]

$$\beta^G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} + 2 D_\mu a_\nu = 0 , \tag{2}$$

$$\beta^B_{\mu\nu} \equiv D_\rho B^\rho{}_{\mu\nu} - 2 a_\rho B^\rho{}_{\mu\nu} = 0 , \tag{3}$$

$$\beta^\Phi \equiv 2\pi\kappa \frac{D-26}{6} - R - \frac{1}{24} B_{\mu\rho\sigma} B^{\mu\rho\sigma} - D_\mu a^\mu + 4a^2 = c , \tag{4}$$

where c is an arbitrary constant. It holds

$$D^\nu \beta^G_{\nu\mu} + \partial_\mu \beta^\Phi = 0 , \tag{5}$$

which means that third beta function, β^Φ, can be zero or nonzero constant. The notation is in the standard form: $R_{\mu\nu}$ and D_μ are Ricci tensor and covariant derivative with respect to the space-time metric $G_{\mu\nu}$, while field strength for Kalb-Ramond field $B_{\mu\nu}$ and dilaton gradient are defined as

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} , \quad a_\mu = \partial_\mu \Phi . \tag{6}$$

Choosing Kalb-Ramond field to be linearly coordinate dependent and dilaton field to be constant, we obtain (2)

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} = 0 . \tag{7}$$

Assuming that Kalb-Ramond field strength is infinitesimal we can take $G_{\mu\nu}$ to be constant but in approximation linear in $B_{\mu\nu\rho}$. As a consequence the third equation (4) becomes

$$2\pi\kappa\frac{D-26}{6}=c\,. \tag{8}$$

The arbitrary constant c can be fixed, $c=-\frac{23\pi}{3}$, which gives $D=3$.

The background fields are of the form

$$G_{\mu\nu}=\begin{pmatrix} R_1^2 & 0 & 0\\ 0 & R_2^2 & 0\\ 0 & 0 & R_3^2\end{pmatrix}\,,\quad B_{\mu\nu}=\begin{pmatrix} 0 & Hz & 0\\ -Hz & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\,, \tag{9}$$

where $R_\mu(\mu=1,2,3)$ are radii of the compact dimensions. If we rescale the coordinates

$$x^\mu\longmapsto x'^\mu=R_\mu x^\mu\,, \tag{10}$$

the form of the metric simplifies

$$G_{\mu\nu}=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\,. \tag{11}$$

After all the action gets the form

$$\begin{aligned} S &= \kappa\int_\Sigma d^2\xi\partial_+x^\mu\Pi_{+\mu\nu}\partial_-x^\nu \qquad (12)\\ &= \kappa\int_\Sigma d^2\xi\left[\frac{1}{2}\left(\partial_+x\partial_-x+\partial_+y\partial_-y+\partial_+z\partial_-z\right)\right.\\ &+ \left.\partial_+xHz\partial_-y-\partial_+yHz\partial_-x\right]\,, \end{aligned}$$

where $\partial_\pm=\partial_\tau\pm\partial_\sigma$, and

$$\Pi_{\pm\mu\nu}=B_{\mu\nu}\pm\frac{1}{2}G_{\mu\nu}\,,\quad x^\mu=\begin{pmatrix} x\\ y\\ z\end{pmatrix}\,. \tag{13}$$

2.2. T-dualization procedure

In the standard Buscher procedure, the starting point is assumption that the target space has isometries. Our background is coordinate dependent so it is useful to examine if the isometry exists. Let us start with the coordinate shift

$$\delta x^\mu=\lambda^\mu=const, \tag{14}$$

and assume that all the coordinates are compact. As $B_{\mu\nu}$ is linear in coordinate, we have

$$\begin{aligned} \delta S &= \frac{\kappa}{3} B_{\mu\nu\rho} \lambda^\rho \int d^2\xi \partial_+ x^\mu \partial_- x^\nu \\ &= \frac{\kappa}{3} B_{\mu\nu\rho} \lambda^\rho \epsilon^{\alpha\beta} \int d^2\xi \partial_\alpha x^\mu \partial_\beta x^\nu. \end{aligned} \tag{15}$$

This is proportional to the total divergence

$$\delta S = \frac{\kappa}{3} B_{\mu\nu\rho} \lambda^\rho \epsilon^{\alpha\beta} \int d^2\xi \partial_\alpha (x^\mu \partial_\beta x^\nu) = 0, \tag{16}$$

which vanishes in the case of the closed string and the topologically trivial mapping of the world-sheet into the space-time. So, the isometry exists even in the case we have chosen.

To localize the global symmetry, we introduce the gauge fields v^μ_α and substitute the ordinary derivatives with the covariant ones

$$\partial_\alpha x^\mu \to D_\alpha x^\mu = \partial_\alpha x^\mu + v^\mu_\alpha. \tag{17}$$

The covariant derivatives are gauge invariant under the following transformation law for the gauge fields

$$\delta v^\mu_\alpha = -\partial_\alpha \lambda^\mu, \quad (\lambda^\mu = \lambda^\mu(\tau, \sigma)). \tag{18}$$

This replacement is not sufficient to make the action locally invariant because the background field $B_{\mu\nu}$ is coordinate dependent. The coordinate x^μ, should be replaced with the invariant coordinate

$$\begin{aligned} \Delta x^\mu_{inv} &\equiv \int_P d\xi^\alpha D_\alpha x^\mu = \int_P (d\xi^+ D_+ x^\mu + d\xi^- D_- x^\nu) \\ &= x^\mu - x^\mu(\xi_0) + \Delta V^\mu, \end{aligned} \tag{19}$$

where

$$\Delta V^\mu \equiv \int_P d\xi^\alpha v^\mu_\alpha = \int_P (d\xi^+ v^\mu_+ + d\xi^- v^\mu_-). \tag{20}$$

In order to make gauge fields v^μ_α nonphysical degrees of freedom, the corresponding field strength

$$F^\mu_{\alpha\beta} \equiv \partial_\alpha v^\mu_\beta - \partial_\beta v^\mu_\alpha, \tag{21}$$

must vanish. Technically, this can be achieved by introducing the Lagrange multiplier y_μ, and the appropriate additional term in the Lagrangian

$$S_{add} = \frac{\kappa}{2} \int d^2 (v^\mu_+ \partial_- y_\mu - v^\mu_- \partial_+ y_\mu), \tag{22}$$

where the last term is equal $\frac{1}{2} y_\mu F^\mu_{+-}$ up to the total divergence. At the end of procedure we fix gauge freedom in the way that $x^\mu(\xi) = x^\mu(\xi_0)$.

3. T-dualization along chain $x \to y \to z$

In this section we will make T-dualization along chain $x \to y \to z$, step by step. Our goal is to find transformation laws, and using them we will calculate noncommutativity and nonassociativity relations.

3.1. Twisted torus geometry from torus with H-flux

Because background fields do not depend on coordinate x, T-dualization along direction x is performed within standard Buscher procedure (without introduction of invariant coordinate). Here we will repeat the standard and generalized Buscher procedure explained above, while in other cases we will just give the final results.

Since x direction is an isometry one, action has a global shift symmetry, $x \longrightarrow x+a$. Localizing this symmetry we replace ordinary derivatives with the covariant ones

$$\partial_\pm x \longrightarrow D_\pm x = \partial_\pm x + v_\pm \,, \tag{23}$$

where $v_\pm$ are gauge field. In order to have T-dual action with the same number of degrees of freedom as initial one, we have to add following term to the action

$$S_{add} = \frac{\kappa}{2}\int_\Sigma d^2\xi y_1(\partial_+ v_- - \partial_- v_+)\,, \tag{24}$$

where y_1 is a Lagrange multiplier. Symmetry enables us to fix gauge, $x = const.$, which produces

$$\begin{aligned} S_{fix} &= \kappa\int d^2\xi \left[\frac{1}{2}\left(v_+ v_- + \partial_+ y\partial_- y + \partial_+ z\partial_- z\right)\right. \\ &+ v_+ Hz\partial_- y - \partial_+ y Hz v_- \\ &+ \left.\frac{1}{2}y_1(\partial_+ v_- - \partial_- v_+)\right]\,. \end{aligned} \tag{25}$$

On the equations of motion for y_1 field strength for the gauge field $v_\pm$ is equal to zero

$$F_{+-} = \partial_+ v_- - \partial_- v_+ = 0\,. \tag{26}$$

Vanishing of the field strength gives us the solution

$$v_\pm = \partial_\pm x\,. \tag{27}$$

Applying this solution from gauge fixed action (25) we restore initial action (12). Varying the gauge fixed action with respect to the gauge fields we get

$$v_- = -\partial_- y_1 - 2Hz\partial_- y\,, \tag{28}$$

$$v_+ = \partial_+ y_1 + 2Hz\partial_+ y\,. \tag{29}$$

Using (28) and (29) from gauge fixed action (25) we get the T-dual action

$$ {}_xS = \kappa \int_\Sigma d^2\xi \partial_+ ({}_xX)^\mu {}_x\Pi_{+\mu\nu} \partial_- ({}_xX)^\nu \,, \tag{30} $$

where

$$ {}_xX^\mu = \begin{pmatrix} y_1 \\ y \\ z \end{pmatrix} , \tag{31} $$

and T-dual background fields

$$ {}_x\Pi_{+\mu\nu} = {}_xB_{\mu\nu} + \frac{1}{2} {}_xG_{\mu\nu} \,, \quad {}_xB_{\mu\nu} = 0 \,, \quad {}_xG_{\mu\nu} = \begin{pmatrix} 1 & 2Hz & 0 \\ 2Hz & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{32} $$

Geometry described by background fields (32) defines so called *twisted torus geometry*. String theory after one T-dualization is geometrical (flux H takes the role of connection).

Combining two sets of equations of motion, (27), (28) and (29), we get the transformation laws

$$ \partial_\pm x \cong \pm \partial_\pm y_1 \pm 2Hz \partial_\pm y \,, \tag{33} $$

where $\cong$ denotes T-duality relation. From the initial action (12) we can easily find canonical momentum π_x

$$ \pi_x = \frac{\delta S}{\delta \dot{x}} = \kappa (\dot{x} - 2Hzy') \,, \tag{34} $$

where $\dot{A} \equiv \partial_\tau A$ and $A' \equiv \partial_\sigma A$. Transformation law (33) produces the relation

$$ \dot{x} \cong y_1' + 2Hzy' \,, \tag{35} $$

which, combined with the expression for π_x, enables us to find transformation law in canonical form

$$ \pi_x \cong \kappa y_1' \,. \tag{36} $$

The initial theory described by action (12) is geometrical one and their coordinates and canonical momenta satisfy standard Poisson algebra

$$ \{x^\mu(\sigma), \pi_\nu(\bar{\sigma})\} = \delta^\mu{}_\nu \delta(\sigma - \bar{\sigma}) \,, \quad \{x^\mu, x^\nu\} = \{\pi_\mu, \pi_\nu\} = 0 \,. \tag{37} $$

Using this algebra it is simply to show that obtained theory (30) is commutative one

$$ \{{}_xX^\mu, {}_xX^\nu\} = 0 \,. \tag{38} $$

3.2. The second step - Q-flux theory

In this subsection the starting action is the action obtained after T-dualization along x (30). Because y is an isometry direction T-dualization along y direction will be performed according to standard Buscher procedure.

Let us construct the gauge fixed action starting with (30)

$$\begin{aligned} S_{fix} &= \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}\left(\partial_+ y_1 \partial_- y_1 + v_+ v_- + \partial_+ z \partial_- z\right)\right. \\ &+ \left. \partial_+ y_1 H z v_- + v_+ H z \partial_- y_1 + \frac{1}{2} y_2 (\partial_+ v_- - \partial_- v_+)\right] . \end{aligned} \tag{39}$$

The equation of motion for Lagrange multiplier y_2 produces

$$\partial_+ v_- - \partial_- v_+ = 0 \longrightarrow v_\pm = \partial_\pm y \,. \tag{40}$$

Application of these equations of motions transfers action (39) to (30). Varying the gauge fixed action with respect to the gauge fields are

$$v_\pm = \pm\partial_\pm y_2 - 2Hz\partial_\pm y_1 \,. \tag{41}$$

Putting these expressions into gauge fixed action, we get T-dual action

$${}_{xy}S = \kappa \int d^2\xi \partial_+ ({}_{xy}X)^\mu {}_{xy}\Pi_{+\mu\nu} \partial_- ({}_{xy}X)^\nu \,, \tag{42}$$

where the background fields are

$${}_{xy}B_{\mu\nu} = \begin{pmatrix} 0 & -Hz & 0 \\ Hz & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -B_{\mu\nu} \,, \quad {}_{xy}G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,, \tag{43}$$

and

$$({}_{xy}X)^\mu = \begin{pmatrix} y_1 \\ y_2 \\ z \end{pmatrix} \,, \quad {}_{xy}\Pi_{+\mu\nu} = {}_{xy}B_{\mu\nu} + \frac{1}{2} {}_{xy}G_{\mu\nu} = \begin{pmatrix} \frac{1}{2} & -Hz & 0 \\ Hz & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} . \tag{44}$$

We see that background fields look like those of torus with H flux (12). Their global properties are non-trivial and that is a reason why the term nongeometry is introduced.

Using T-dual transformation laws in Lagrangian form

$$\partial_\pm y \cong \pm\partial_\pm y_2 - 2Hz\partial_\pm y_1 \,, \tag{45}$$

in combination with the expression for canonical momentum of the initial theory

$$\pi_y = \frac{\delta S}{\delta \dot{y}} = \kappa(\dot{y} + 2Hzx') \,. \tag{46}$$

we get T-dual transformation law in canonical form

$$\pi_y \cong \kappa y_2' \,. \tag{47}$$

From standard Poisson algebra (37) it follows that theory obtained by two T-dualizations along isometry directions is still commutative

$$\{{}_{xy}X^\mu, {}_{xy}X^\nu\} = 0\,. \tag{48}$$

3.3. Full T-dualized theory

Kalb-Ramond field is dependent on z but this direction is still isometry one (see subsection 2.2). For T-dualization along z we use generalized T-dualization procedure [12, 13, 14].

Starting action is the one obtained in the previous step (42). The first step in the T-dualization procedure is localizing shift symmetry of the action (42) along z direction. This means that we have to introduce covariant derivative

$$\partial_\pm z \longrightarrow D_\pm z = \partial_\pm z + v_\pm \,. \tag{49}$$

Then we introduce the invariant coordinate as line integral

$$z^{inv} = \int_P d\xi^\alpha D_\alpha z = \int_P d\xi^+ D_+ z + \int_P d\xi^- D_- z = z(\xi) - z(\xi_0) + \Delta V\,, \tag{50}$$

where

$$\Delta V = \int_P d\xi^\alpha v_\alpha = \int_P (d\xi^+ v_+ + d\xi^- v_-)\,. \tag{51}$$

In order to make $v_\pm$ to be nonphysical degrees of freedom we add to the action term with Lagrange multiplier

$$S_{add} = \frac{\kappa}{2}\int_\Sigma d^2\xi \; y_3(\partial_+ v_- - \partial_+ v_-)\,. \tag{52}$$

The final form of the action is

$$\begin{aligned}\bar{S} &= \kappa \int_\Sigma d^2\xi \Big[-H z^{inv}(\partial_+ y_1 \partial_- y_2 - \partial_+ y_2 \partial_- y_1) \\ &+ \frac{1}{2}(\partial_+ y_1 \partial_- y_1 + \partial_+ y_2 \partial_- y_2 + D_+ z D_- z) + \frac{1}{2}\, y_3(\partial_+ v_- - \partial_- v_+)\Big]\,.\end{aligned} \tag{53}$$

The final step in the procedure is gauge fixing, $z(\xi) = z(\xi_0)$, and then the gauge fixed action is of the form

$$\begin{aligned}S_{fix} &= \kappa \int_\Sigma d^2\xi \,[-H\Delta V(\partial_+ y_1 \partial_- y_2 - \partial_+ y_2 \partial_- y_1) \\ &+ \frac{1}{2}(\partial_+ y_1 \partial_- y_1 + \partial_+ y_2 \partial_- y_2 + v_+ v_-) + \frac{1}{2} y_3(\partial_+ v_- - \partial_- v_+)\Big]\,.\end{aligned} \tag{54}$$

We restore initial theory (42) from the gauge fixed action using equation of motion for Lagrange multiplier y_3

$$\partial_+ v_- - \partial_- v_+ = 0 \Longrightarrow v_\pm = \partial_\pm z\,, \quad \Delta V = \Delta z\,. \tag{55}$$

Varying the gauge fixed action (54) with respect to the gauge fields we get

$$v_\pm = \pm\partial_\pm y_3 - 2\beta^\mp\,, \tag{56}$$

where $\beta^\pm$ are functions defined as

$$\beta^\pm = \pm\frac{1}{2}H(y_1\partial_\mp y_2 - y_2\partial_\mp y_1)\,. \tag{57}$$

They are a result of the variation of the term containing ΔV. Using relations (56) from the gauge fixed action, we obtain the T-dual action

$${}_{xyz}S = \kappa\int_\Sigma d^2\xi\,\partial_{+\,xyz}X^\mu\,{}_{xyz}\Pi_{+\mu\nu}\partial_{-\,xyz}X^\nu\,, \tag{58}$$

where background fields are

$${}_{xyz}B_{\mu\nu} = \begin{pmatrix} 0 & -H\Delta\tilde{y}_3 & 0 \\ H\Delta\tilde{y}_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \quad {}_{xyz}G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,, \tag{59}$$

and

$${}_{xyz}X^\mu = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\,. \tag{60}$$

The double coordinate $\tilde{y}_3$ is defined as

$$\partial_\pm y_3 \equiv \pm\partial_\pm\tilde{y}_3\,, \tag{61}$$

while, because it stands together with H, we calculate ΔV in the zeroth order

$$\Delta V = \int d\xi^+\partial_+ y_3 - \int d\xi^-\partial_- y_3\,. \tag{62}$$

Because ΔV is defined as line integral, the nonlocality occures in the T-dual theory. The result of the three T-dualization is a theory with R flux.

Lagrangian form of the T-dual transformation law is

$$\partial_\pm z \cong \pm\partial_\pm y_3 - 2\beta^\mp\,. \tag{63}$$

while its canonical form is

$$y_3' \cong \frac{1}{\kappa}\pi_z - H(xy' - yx')\,. \tag{64}$$

4. Noncommutativity and nonassociativity

In this section we will calculate noncommutativity and nonassociativity relations using canonical forms of the transformation laws.

4.1. Noncommutativity relations

Rewriting the relations (36), (47) and (64) as

$$y_1' \cong \frac{1}{\kappa}\pi_x\,, \quad y_2' \cong \frac{1}{\kappa}\pi_y\,, \quad y_3' \cong \frac{1}{\kappa}\pi_z - H(xy' - yx')\,. \tag{65}$$

we conclude that nontrivial Poisson brackets will be $\{y_1(\sigma), y_3(\bar\sigma)\}$ and $\{y_2(\sigma), y_3(\bar\sigma)\}$. Using the result presented in the Appendix A and the relations

$$\{y_1'(\sigma), y_3'(\bar\sigma)\} \cong \frac{2}{\kappa}Hy'(\sigma)\delta(\sigma-\bar\sigma) + \frac{1}{\kappa}Hy(\sigma)\delta'(\sigma-\bar\sigma)\,, \tag{66}$$

$$\{y_2'(\sigma), y_3'(\bar\sigma)\} \cong -\frac{2}{\kappa}Hx'(\sigma)\delta(\sigma-\bar\sigma) - \frac{1}{\kappa}Hx(\sigma)\delta'(\sigma-\bar\sigma)\,, \tag{67}$$

we get the Poisson brackets of the T-dual coordinates

$$\{y_1(\sigma), y_3(\bar\sigma)\} \cong -\frac{H}{\kappa}\left[2y(\sigma) - y(\bar\sigma)\right]\theta(\sigma-\bar\sigma)\,, \tag{68}$$

$$\{y_2(\sigma), y_3(\bar\sigma)\} \cong \frac{H}{\kappa}\left[2x(\sigma) - x(\bar\sigma)\right]\theta(\sigma-\bar\sigma)\,. \tag{69}$$

If $\sigma = \bar\sigma$ then these two Poisson brackets are zero. But if we choose that $\sigma - \bar\sigma = 2\pi$ then $\theta(2\pi) = 1$ and it follows

$$\{y_1(\sigma+2\pi), y_3(\sigma)\} \cong -\frac{H}{\kappa}\left[4\pi N_y + y(\sigma)\right]\,, \tag{70}$$

$$\{y_2(\sigma+2\pi), y_3(\sigma)\} \cong \frac{H}{\kappa}\left[4\pi N_x + x(\sigma)\right]\,, \tag{71}$$

where N_x and N_y are winding numbers defined as

$$x(\sigma+2\pi) - x(\sigma) = 2\pi N_x\,, \quad y(\sigma+2\pi) - y(\sigma) = 2\pi N_y\,. \tag{72}$$

4.2. Nonassociativity

Let us start calculating Poisson brackets $\{y_1(\sigma), x(\bar\sigma)\}$ and $\{y_2(\sigma), y(\bar\sigma)\}$. Similar to the calculations presented in Appendix A, we start with

$$\{\Delta y_1(\sigma,\sigma_0), x(\bar\sigma)\} = \{\int_{\sigma_0}^{\sigma} d\eta\, y_1'(\eta), x(\bar\sigma)\}\,. \tag{73}$$

Using the T-dual transformation in canonical form (36), we obtain

$$\{\Delta y_1(\sigma,\sigma_0), x(\bar\sigma)\} \cong \frac{1}{\kappa}\{\int_{\sigma_0}^{\sigma} d\eta \pi_x(\eta), x(\bar\sigma)\}\,, \tag{74}$$

which produces further

$$\begin{aligned} & \{\Delta y_1(\sigma,\sigma_0), x(\bar\sigma)\} \cong -\frac{1}{\kappa}\left[\theta(\sigma-\bar\sigma) - \theta(\sigma_0-\bar\sigma)\right] \\ \Longrightarrow \quad & \{y_1(\sigma), x(\bar\sigma)\} \cong -\frac{1}{\kappa}\theta(\sigma-\bar\sigma)\,. \end{aligned} \tag{75}$$

The relation $\{y_2(\sigma), y(\bar\sigma)\}$ can be obtained in the same way

$$\{y_2(\sigma), y(\bar\sigma)\} \cong -\frac{1}{\kappa}\theta(\sigma-\bar\sigma)\,. \tag{76}$$

Now it is straightforward to calculate Jacobi identity using (68), (69) and above Poisson brackets

$$\begin{aligned} & \{y_1(\sigma_1), y_2(\sigma_2), y_3(\sigma_3)\} \equiv \\ & \{y_1(\sigma_1), \{y_2(\sigma_2), y_3(\sigma_3)\}\} + \{y_2(\sigma_2), \{y_3(\sigma_3), y_1(\sigma_1)\}\} \\ + \quad & \{y_3(\sigma_3), \{y_1(\sigma_1), y_2(\sigma_2)\}\} \cong \\ & -\frac{2H}{\kappa^2}\left[\theta(\sigma_1-\sigma_2)\theta(\sigma_2-\sigma_3) + \theta(\sigma_2-\sigma_1)\theta(\sigma_1-\sigma_3)\right. \\ + \quad & \left.\theta(\sigma_1-\sigma_3)\theta(\sigma_3-\sigma_2)\right]\,. \end{aligned} \tag{77}$$

Jacobi identity is nonzero which means that theory with R-flux is in general nonassociative. For $\sigma_2 = \sigma_3 = \sigma$ and $\sigma_1 = \sigma + 2\pi$ we get

$$\{y_1(\sigma+2\pi), y_2(\sigma), y_3(\sigma)\} \cong \frac{2H}{\kappa^2}\,. \tag{78}$$

5. Reversed T-dualization chain $z \to y \to x$

Our intention is to T-dualize (12) in opposite direction to find the form of noncommutativity and nonassociativity relations. Because we T-dualize first along z, all three theories are R-flux ones.

The T-dualization procedure is presented and already applied three times. So, omitting all technical steps, the T-dual action is of the form

$${}_zS = \kappa \int_\Sigma d^2\xi \partial_+ {}_zX^\mu {}_z\Pi_{+\mu\nu} \partial_- {}_zX^\nu\,, \tag{79}$$

where

$${}_zX^\mu = \begin{pmatrix} x \\ y \\ y_3 \end{pmatrix}, \quad {}_z\Pi_{+\mu\nu} = {}_zB_{\mu\nu} + \frac{1}{2}{}_zG_{\mu\nu}\,, \tag{80}$$

$$
{}_zB_{\mu\nu} = \begin{pmatrix} 0 & H\Delta V & 0 \\ -H\Delta V & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad {}_zG_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{81}
$$

T-dual transformation laws are

$$
\partial_\pm z \cong \pm\partial_\pm y_3 \mp H(x\partial_\pm y - y\partial_\pm x). \tag{82}
$$

Momentum of the initial theory (12) canonically conjugated to the coordinate z is of the form

$$
\pi_z = \kappa\dot{z}, \tag{83}
$$

so, the T-dual transformation law in canonical form is

$$
y_3' \cong \frac{1}{\kappa}\pi_z + H(xy' - yx'). \tag{84}
$$

From the expressions (84) and (37), we get that coordinates ${}_zX^\mu$ are commutative. Consequently, Jacobiator is equal to zero, which means that theory is associative.

After T-dualization along y direction the T-dual action is

$$
{}_{zy}S = \kappa\int_\Sigma d^2\xi\; \partial_+\, {}_{zy}X^\mu\; {}_{zy}\Pi_{+\mu\nu}\; \partial_-\, {}_{zy}X^\nu, \tag{85}
$$

where

$$
{}_{zy}X^\mu = \begin{pmatrix} x \\ y_2 \\ y_3 \end{pmatrix}, \quad {}_{zy}\Pi_{+\mu\nu} = {}_{zy}B_{\mu\nu} + \frac{1}{2}\, {}_{zy}G_{\mu\nu}, \tag{86}
$$

$$
{}_{zy}B_{\mu\nu} = 0, \quad {}_{zy}G_{\mu\nu} = \begin{pmatrix} 1 & -2H\Delta V & 0 \\ -2H\Delta V & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{87}
$$

The canonical form of the T-dual transformation law is

$$
y_2' \cong \frac{1}{\kappa}\pi_y. \tag{88}
$$

The only non-zero Poisson bracket is

$$
\{y_2'(\sigma), y_3'(\bar{\sigma})\} \cong \frac{H}{\kappa}\left[2x'(\sigma)\delta(\sigma-\bar{\sigma}) + x(\sigma)\delta'(\sigma-\bar{\sigma})\right]. \tag{89}
$$

Using the instructions from Appendix A, the solution is of the form (107)

$$
\{y_2(\sigma), y_3(\bar{\sigma})\} \cong -\frac{H}{\kappa}\left[2x(\sigma) - x(\bar{\sigma})\right]\theta(\sigma-\bar{\sigma}). \tag{90}
$$

For $\sigma - \bar{\sigma} = 2\pi$, we have

$$\{y_2(\sigma + 2\pi), y_3(\sigma)\} \cong -\frac{H}{\kappa}\left[x(\sigma) + 4\pi N_x\right] , \tag{91}$$

where N_x is winding number for x coordinate defined as

$$x(\sigma + 2\pi) - x(\sigma) = 2\pi N_x \,. \tag{92}$$

It is straightforward to calculate the Jacobiator

$$\begin{aligned} &\{x(\sigma_1), \{y_2(\sigma_2), y_3(\sigma_3)\}\} + \{y_2(\sigma_2), \{y_3(\sigma_3), x(\sigma_1)\}\} \\ + \;\; &\{y_3(\sigma_3), \{x(\sigma_1), y_2(\sigma_2)\}\} \cong 0 \,. \end{aligned} \tag{93}$$

Because it is zero, we conclude that this R-flux theory is noncommutative and associative one.

The full T-dualozed action is

$$_{zyx}S = \kappa \int_\Sigma d^2\xi \partial_+ \, _{zyx}X^\mu \, _{zyx}\Pi_{+\mu\nu} \, _{zyx} \, X^\nu \,, \tag{94}$$

where

$$_{zyx}X^\mu = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad _{zyx}\Pi_{+\mu\nu} = \, _{zyx}B_{\mu\nu} + \frac{1}{2} \, _{zyx}G_{\mu\nu} \tag{95}$$

$$_{zyx}B_{\mu\nu} = \begin{pmatrix} 0 & -H\Delta V & 0 \\ H\Delta V & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad _{zyx}G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{96}$$

The canonical form of the T-dual transformation law is the same as in the case $x \to y \to z$

$$y_1' \cong \frac{1}{\kappa}\pi_x \,. \tag{97}$$

The set of T-dual transformation laws, (84), (88) and (97), is the same as in the xyz case up to $H \to -H$. Consequently, we have

$$\{y_1(\sigma), y_3(\bar{\sigma})\} \cong \frac{H}{\kappa}\left[2y(\sigma) - y(\bar{\sigma})\right]\theta(\sigma - \bar{\sigma}) \,, \tag{98}$$

$$\{y_2(\sigma), y_3(\bar{\sigma})\} \cong -\frac{H}{\kappa}\left[2x(\sigma) - x(\bar{\sigma})\right]\theta(\sigma - \bar{\sigma}) \,, \tag{99}$$

and nonassociativity

$$\begin{aligned} &\{y_1(\sigma_1), y_2(\sigma_2), y_3(\sigma_3)\} \equiv \\ &\{y_1(\sigma_1), \{y_2(\sigma_2), y_3(\sigma_3)\}\} + \{y_2(\sigma_2), \{y_3(\sigma_3), y_1(\sigma_1)\}\} \\ + \;\; &\{y_3(\sigma_3), \{y_1(\sigma_1), y_2(\sigma_2)\}\} \cong \\ &\frac{2H}{\kappa^2}\left[\theta(\sigma_1 - \sigma_2)\theta(\sigma_2 - \sigma_3) + \theta(\sigma_2 - \sigma_1)\theta(\sigma_1 - \sigma_3)\right. \\ + \;\; &\left.\theta(\sigma_1 - \sigma_3)\theta(\sigma_3 - \sigma_2)\right] . \end{aligned} \tag{100}$$

6. Conclusion

In this article we considered closed bosonic string propagating in the constant metric and Kalb-Ramond field with just one nonzero component, $B_{xy} = -B_{yx} = Hz$. We worked in the approximation linear in H. We T-dualized along chain xyz and in opposite direction. Our goal was to examine the influence of the sequence of T-dualizations on the full T-dualized theory, geometrical features and noncommutativity and nonassociativity of the coordinates.

T-dualizing along chain xyz we concluded that the first theory is geometrical, commutative and associative, the second one is nongeometrical, commutative and associative, while the full T-dualized theory is nongeometrical R flux theory which is noncommuattive and nonassociative. When we reversed the direction of T-dualizations we obtained that all three theories are non geometrical R flux ones. But the first one is both commutative and associative, the second one is noncommutative and associative, while the third one is both nonassociative and noncommutative. The corresponding parameters can be obtained from those in xyz case by replacing $H \to -H$. The form of the full T-dualized theory does not depend on the sequence of T-dualizations.

A Important Poisson bracket

Let us start with the Poisson bracket of the σ derivatives of two arbitrary functions in the form

$$\{A'(\sigma), B'(\bar{\sigma})\} = U'(\sigma)\delta(\sigma - \bar{\sigma}) + V(\sigma)\delta'(\sigma - \bar{\sigma}) \,, \tag{101}$$

where $\delta'(\sigma - \bar{\sigma}) \equiv \partial_\sigma \delta(\sigma - \bar{\sigma})$. Our task is to find

$$\{A(\sigma), B(\bar{\sigma})\} \,,$$

from

$$\{\Delta A(\sigma, \sigma_0), \Delta B(\bar{\sigma}, \bar{\sigma}_0)\} \,,$$

where

$$\Delta A(\sigma, \sigma_0) = \int_{\sigma_0}^{\sigma} dx A'(x) = A(\sigma) - A(\sigma_0) \,,$$
$$\Delta B(\bar{\sigma}, \bar{\sigma}_0) = \int_{\bar{\sigma}_0}^{\bar{\sigma}} dx B'(x) = B(\bar{\sigma}) - B(\bar{\sigma}_0) \,. \tag{102}$$

It is obvious that

$$\{\Delta A(\sigma, \sigma_0), \Delta B(\bar{\sigma}, \bar{\sigma}_0)\} = \int_{\sigma_0}^{\sigma} dx \int_{\bar{\sigma}_0}^{\bar{\sigma}} dy \ \left[U'(x)\delta(x - y) + V(x)\delta'(x - y)\right] \,, \tag{103}$$

and after integration over y we get

$$\begin{aligned} & \{\Delta A(\sigma,\sigma_0), \Delta B(\bar\sigma,\bar\sigma_0)\} \\ = & \int_{\sigma_0}^{\sigma} dx \{U'(x)\,[\theta(x-\bar\sigma_0)-\theta(x-\bar\sigma)] + V(x)\,[\delta(x-\bar\sigma_0)-\delta(x-\bar\sigma)]\}, \end{aligned} \tag{104}$$

where $\theta(x)$ is defined as

$$\theta(x) = \int_0^x d\eta\delta(\eta) = \frac{1}{2\pi}\left[x + 2\sum_{n\geq 1}\frac{1}{n}\sin(nx)\right] = \begin{cases} 0 & \text{if } x=0 \\ 1/2 & \text{if } 0<x<2\pi \\ 1 & \text{if } x=2\pi \end{cases}. \tag{105}$$

Integrating over x, finally we get

$$\begin{aligned} & \{\Delta A(\sigma,\sigma_0), \Delta B(\bar\sigma,\bar\sigma_0)\} = \\ & U(\sigma)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma-\bar\sigma)] - U(\sigma_0)[\theta(\sigma_0-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma)] \\ - & U(\bar\sigma_0)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma_0)] + U(\bar\sigma)[\theta(\sigma-\bar\sigma)-\theta(\sigma_0-\bar\sigma)] \\ + & V(\bar\sigma_0)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma_0] - V(\bar\sigma)[\theta(\sigma-\bar\sigma)-\theta(\sigma_0-\bar\sigma)] \end{aligned} \tag{106}$$

From the last expression, we extract the desired Poisson bracket

$$\{A(\sigma), B(\bar\sigma)\} = -[U(\sigma)-U(\bar\sigma)+V(\bar\sigma)]\theta(\sigma-\bar\sigma)\,. \tag{107}$$

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