Polynomial Representations of the Lie superalgebra $\mathfrak{osp}(1|2n)$

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18. June 2019

Joint work with Joris Van der Jeugt and Hendrik de Bie

Representation Theory of $\mathfrak{osp}(1|2n)$

- Classify the representations.
- Find character formulas.
- Construct bases.
- Calculate matrix elements.
- Formulate inner products.

What do we know about $\mathfrak{osp}(1|2n)$ representations?:

- Finite dimensional representations:
  - Classified and characters are understood. [Kac; 1977]
- Infinite dimensional representations:
  - Lowest weight representations: Classified and characters are mostly understood. [Dobrev, Zhang; 2006].
  - Paraboson Fock representations: Character formula, basis and matrix elements("abstract"). [Lievens, Stoilova, Van der Jeugt; 2008]

Definition of $\mathfrak{osp}(1|2n)$

- Definition as a matrix algebra [Kac; 1977].
- Definition as a superalgebra by means of generators and relations [Ganchev, Palev; 1980]:
  Odd generators

$$b_i^+, b_i^-, \quad i \in \{1, \ldots, n\},$$

satisfying

$$[\{b_i^\xi, b_j^\eta\}, b_l^\epsilon] = (\epsilon - \xi)\delta_{i,l} b_j^\eta + (\epsilon - \eta)\delta_{j,l} b_i^\xi,$$

for $i, j, l \in \{1, \ldots, n\}$ and $\eta, \epsilon, \xi \in \{+, -\}$, to be interpreted as ± 1 in the algebraic relations.

We can interpret b_i^+ and b_i^- as parabosonic creation and annihilation operators.

Parabosonic Fock space

Definition 1

For $p \in \mathbb{N}$, the paraboson Fock space is an $\mathfrak{osp}(1|2n)$ irrep. with unique vacuum $|0\rangle$ satisfying

$$b_i^+|0\rangle = 0 \text{ and } \frac{1}{2}\{b_i^-, b_i^+\}|0\rangle = \frac{p}{2}|0\rangle, \quad (i \in \{1, \dots, n\}).$$

$\mathbb{C}[\mathbb{R}^{np}]$ polynomials in $n \cdot p$ variables, $x_{i,j}$,

$\mathcal{C}\ell_p$ Clifford algebra generated by e_j, satisfying $\{e_i, e_j\} = 2\delta_{ij}$.

Then $\mathfrak{osp}(1|2n)$ acts on $\mathbb{C}[\mathbb{R}^{np}] \otimes \mathcal{C}\ell_p$ with,

$$b_i^+ \mapsto X_i = \sum_{j=1}^{p} \underbrace{x_{i,j} e_j,}_{\text{Green's ansatz}} \quad \text{and } b_i^- \mapsto D_i = \sum_{j=1}^{p} \underbrace{\partial_{x_{i,j}} e_j.}_{\text{Green's ansatz}}$$

Polynomial Representation

Let $W(\mu + \frac{p}{2})$ be the $\mathfrak{osp}(1|2n)$ irrep. with a (not necessarily unique) vacuum vector $|\mu; 0\rangle$ satisfying

$$\frac{1}{2}\{b_i^-, b_i^+\}|\mu; 0\rangle = (\mu + \frac{p}{2})|\mu; 0\rangle, \quad (i \in \{1, \dots, n\}).$$

Decomposition [Cheng, Kwong, Wang; 2010], [Salom; 2013]:

$$\mathbb{C}[\mathbb{R}^{np}] \otimes \mathcal{C}\ell_p = \bigoplus_{\mu \in \mathcal{P}} m_{\mu + \frac{p}{2}} W(\mu + \frac{p}{2}),$$

with $m_{\mu + \frac{p}{2}}$ being the multiplicities.
The case $\mu = 0$ gives a paraboson Fock space. Let $|0; 0\rangle \mapsto 1$, then

$$\frac{1}{2}\{D_i, X_i\}(1) = \frac{p}{2}, \quad (i \in \{1, \dots, n\}),$$

$$W(\frac{p}{2}) = \text{span}_{\mathbb{C}}\{X_1^{k_1} \cdots X_n^{k_n}(1) : k_1, \dots, k_n \in \mathbb{N}_0\}.$$

Character Formula

$\mathcal{P}$ Set of all partitions. $(\lambda_1, \ldots, \lambda_k), \lambda_1 \geq \cdots \geq \lambda_k, k \in \mathbb{N}_0$

$\ell(\lambda)$ Length of $\lambda \in \mathcal{P}$. $\lambda_{\ell(\lambda)} > 0, \lambda_l = 0, l > \ell(\lambda)$

s_λ Schur function of the partition $\lambda \in \mathcal{P}$

$K_{\lambda\mu}$ Kostka numbers for $\lambda \in \mathcal{P}$ and $\mu \in \mathbb{N}_0^n$

Theorem (Lievens, Stoilova, Van der Jeugt; 2008)

$$\begin{aligned} \operatorname{char} W(\frac{p}{2}) &= (t_1 \cdots t_n)^{p/2} \sum_{\lambda \in \mathcal{P},\ \ell(\lambda) \leq p} s_\lambda(t_1, \ldots, t_n) \\ &= (t_1 \cdots t_n)^{p/2} \sum_{\lambda \in \mathcal{P}, \ell(\lambda) \leq p} \sum_{\mu \in \mathbb{N}_0^n} K_{\lambda\mu} t_1^{\mu_1} \cdots t_n^{\mu_n} \end{aligned}$$

Weight Spaces

Definition

$$\operatorname{char} W(\frac{p}{2}) = (t_1 \cdots t_n)^{p/2} \sum_{\mu \in \mathbb{N}_0^n} \dim W(\frac{p}{2})_{\mu + \frac{p}{2}} \; t_1^{\mu_1} \cdots t_n^{\mu_n}.$$

So for all $\mu \in \mathbb{N}_0^n$

$$\dim W(\frac{p}{2})_{\mu + \frac{p}{2}} = \sum_{\lambda \in \mathcal{P}, \ell(\lambda) \leq p} K_{\lambda\mu}$$

$K_{\lambda\mu} := \#\{\mathit{Semistandard\ Young\ Tableaux\ of\ shape}\ \lambda\ \mathit{and\ weight}\ \mu\}$

$$< \left\{ \overbrace{\begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 4 & \\ \cline{1-3} 5 & 5 & & \\ \cline{1-2} \end{array}}^{\leq} \right. , \quad \lambda = (4,3,2), \text{ and } \mu = (2,2,1,2,2)$$

Young Tableaux and Basis

$$\mathbb{Y}(p) = \left\{ \begin{matrix} \textit{S.s. Young Tableaux of at most p rows,} \\ \text{and weight in } \mathbb{N}_0^n \end{matrix} \right\}$$

There exists a basis for $W(\frac{p}{2})$:

- Consisting of vectors ν_A, for $A \in \mathbb{Y}(p)$.
- Tableaux $A \in \mathbb{Y}(p)$ of weight μ, gives

$$\nu_A \in W(\frac{p}{2})_{\mu + \frac{p}{2}}.$$

Tableaux Vectors

$$\begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \vdots \\ \hline i_k \\ \hline \end{array} \quad \to I = (i_1, \ldots, i_k) \to X_I := \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) X_{i_{\sigma(1)}} \cdots X_{i_{\sigma(k)}}$$

Definition

For $A = (A[1], \ldots, A[l]) \in \mathbb{Y}(p)$, s.s. Young tableau with l columns, define

$$\omega_A := X_{A[l]} X_{A[l-1]} \cdots X_{A[1]}(1).$$

Remark

For each I, $X_I = 0$ iff $k > p$.

Basis Vectors

$$A = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 5 & 5 \\ \cline{1-2}\end{array} \mapsto \omega_A = X_4 X_{(2,4)} X_{(1,3,5)} X_{(1,2,5)}(1).$$

Theorem

$W(\frac{p}{2})$ has basis

$$\{\omega_A \in W(\frac{p}{2}) : A \in \mathbb{Y}(p)\},$$

with $\omega_A \in W(\frac{p}{2})_{\mu_A + \frac{p}{2}}$.

Proof strategy:
Construct a total order $<$ on $\mathbb{Y}(p)$ such that

$$\omega_A \notin \operatorname{span}\{\omega_B : B \in \mathbb{Y}(p), B < A\}.$$

Monomial Expansion

$M_{n,p}(\mathbb{N}_0)$, n by p positive integer matrices

$\gamma \in M_{n,p}(\mathbb{N}_0)$

$$\mu_\gamma = \sum_{j=1}^{p} (\mu_{1,j}, \ldots, \mu_{n,j}), \quad \text{and} \quad \eta_\gamma = \sum_{i=1}^{n} (\mu_{i,1}, \ldots, \mu_{i,p})$$

$$x^\gamma = \prod_{i=1}^{n} \prod_{j=1}^{p} x_{i,j}^{\gamma_{i,j}} \quad \text{and} \quad e^{\eta_\gamma} = e_1^{(\eta_\gamma)_1} \cdots e_p^{(\eta_\gamma)_p}.$$

Proposition

For $A \in \mathbb{Y}(p)$ of shape $\lambda_A \in \mathcal{P}$,

$$\omega_A = (\lambda_A)'_1! \cdots (\lambda_A)'_{(\lambda_A)_1}! \sum_{\substack{\gamma \in M_{n,p}(\mathbb{N}_0) \\ \mu_A = \mu_\gamma}} c_A(\gamma) x^\gamma e^{\eta_\gamma}$$

Monomial Coefficients

$$\color{blue}{A^k} \in \mathbb{Y}(p) \ k\text{'th subtableaux of } A \in \mathbb{Y}(p),$$

$$A = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 4 & \\ \cline{1-3}\end{array} \quad \Longrightarrow \quad A^4 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 4 & \\ \cline{1-3}\end{array}, A^3 = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \cline{1-2}\end{array}, A^2 = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & & \\ \cline{1-1}\end{array},$$

Theorem

Let $A \in \mathbb{Y}(p)_n(p)$, $\sigma \in S_{\mu_A}$ and $\gamma \in M_{n,p}(\mathbb{N}_0)$ with $\mu_\gamma = \mu_A$.

$$c_A(\gamma) = \frac{1}{(\eta_\gamma)_1! \cdots (\eta_\gamma)_p!} \sum_{\sigma \in S_{\mu_A}} \operatorname{sgn}(\sigma)(-1)^{N_A(\gamma)} \prod_{\alpha=1}^{p} \operatorname{sgn}(L_A(\sigma, \alpha))$$

The values $N_A(\gamma)$ and $L_A(\sigma, \alpha)$ being combinatorial expressions in

$$\color{blue}{\gamma} \in M_{n,p}(\mathbb{N}_0), \color{blue}{\lambda_{A^1}, \ldots, \lambda_{A^n}} \in \mathcal{P} \text{ and } \color{blue}{\sigma} \in S_{(\mu_A)_1} \times \cdots \times S_{(\mu_A)_n}$$

Leading Monomial

Proposition

For $A, B \in \mathbb{Y}(p)$, $B < A$, then $c_A(\gamma) \in \mathbb{Z}$ for all $\gamma \in M_{n,p}(\mathbb{N}_0)$ and

a) $c_A(\gamma_A) \neq 0$

b) $c_B(\gamma_A) = 0$

Where

$$\begin{aligned}(\gamma_A)_{i,j} &= \#\{\textit{Number of } i\textit{'s in the } j\textit{'th row of } A\} \\ &= (\lambda_{A^i})_j - (\lambda_{A^{i-1}})_j\end{aligned}$$

$$d_\mu := \dim W(\frac{p}{2})_{\mu+\frac{p}{2}}.$$

$\{A_1, \ldots, A_{d_\mu}\} \subset \mathbb{Y}(p)$, tableaux of weight μ s.t.

$$A_1 < \cdots < A_{d_\mu}.$$

Action on Basis Elements

Inner product on $W(\frac{p}{2}) \subset \mathbb{C}[\mathbb{R}^{np}] \otimes \mathcal{C}\ell_p$:

$$\langle x^{\gamma} e^{\eta}, x^{\gamma'} e^{\eta'} \rangle := \delta_{\gamma,\gamma'} \delta_{\eta,\eta'}.$$

For $v \in W(\frac{p}{2})$ and $k, l \in \{1, \dots, d_\mu\}$,

$$(U_\mu)_{k,l} = c_{A_k}(\gamma_{A_l}), \quad \text{and} \quad f_\mu(v)_l = \langle x^{\gamma_{A_l}} e^{\eta_{\gamma_{A_l}}}, v \rangle$$

$$\overline{\omega}_B = \frac{1}{(\lambda_B)_1! \cdots (\lambda_B)_n!} \omega_B.$$

Proposition

The matrix U_μ is integer and upper triangular, and for any $v \in \dim(W_n(p))_{\mu + \frac{p}{2}}$,

$$v = \sum_{k=1}^{d_\mu} (U_\mu^{-1}(f_\mu(v))_k \overline{\omega}_{A_l}.$$

Action on Basis Elements

$$X_i\overline{\omega}_A \in W_n(p)_{\mu_A+\epsilon_i+\frac{p}{2}}, \quad \text{and} \quad D_i\overline{\omega}_A \in W_n(p)_{\mu_A-\epsilon_i+\frac{p}{2}}.$$

Proposition

Let $A, B \in \mathbb{Y}(p)$ and $i \in \{1, \dots, n\}$. Then

$$\langle x^{\gamma_A} e^{\eta\gamma_A}, X_i\bar{\omega}_B(p)\rangle = \sum_{\alpha=1}^{p} \prod_{\beta=1}^{\alpha-1} (-1)^{(\lambda_A)_\beta} c_B(\gamma_A - \epsilon_{i,\alpha})$$

$$\langle x^{\gamma_A} e^{\eta\gamma_A}, D_i\bar{\omega}_B(p)\rangle = \sum_{\alpha=1}^{p} \prod_{\beta=1}^{\alpha-1} (-1)^{(\lambda_A)_\beta} ((\gamma_A)_{i,\alpha} + 1) c_B(\gamma_A + \epsilon_{i,\alpha}).$$

This determines the vector $f_\mu(X_i\omega_B)$ and $f_\mu(D_i\omega_B)$, and thus the action.

Example

We calculate $X_i \omega_B$ for $B = \ytableaushort{23,4}$, $\mu_B = (0,1,1,1)$, $i = 1$:

$$A_1 = \ytableaushort{1,2,3,4}, \quad A_2 = \ytableaushort{14,2,3}, \quad A_3 = \ytableaushort{13,2,4}, \quad A_4 = \ytableaushort{13,24}, \quad A_5 = \ytableaushort{134,2},$$

$$A_6 = \ytableaushort{12,3,4}, \quad A_7 = \ytableaushort{12,34}, \quad A_8 = \ytableaushort{124,3}, \quad A_9 = \ytableaushort{123,4}, \quad A_{10} = \ytableaushort{1234}.$$

$$U_{\mu_B+\epsilon_i} = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad f_{\mu_B+\epsilon_i} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ -1 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

$$X_1 \omega_{\ytableaushort{23,4}} = -8\omega_{A_1} - 4\omega_{A_2} + 3\omega_{A_3} - 2\omega_{A_4} - 1\omega_{A_5} - 4\omega_{A_6} + 2\omega_{A_7} + \omega_{A_9}.$$