Extended geometry and tensor hierarchy algebras

Martin Cederwall

Abstract: Extended geometry is a unified framework for double geometry, exceptional geometry, etc. In the talk, I will explain the structure of gauge transformations (generalised diffeomorphisms) in these models. They are generically infinitely reducible, and arise as derived brackets from an underlying tensor hierarchy algebra. The infinite reducibility gives rise to an L_∞ structure, the brackets of which have universal expressions in terms of the underlying superalgebra. Dynamics is formulated in terms of the auperalgebra.

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MC, Palmkvist, 1711.07694
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Part of the motivation: the appearances of dualities in string theory / M-theory.

When M-theory is compactified on an n-torus, the U-duality group is $E_{n(n)}(\mathbb{Z})$.

The duality mixes momenta and brane windings. If this symmetry is to be "geometrised", also diffeomorphisms and tensor gauge transformations need to be unified.

Membranes can wind on 2-cycles,
5-branes on 5-cycles.

An example: A 6-dimensional torus. T^6

$$\binom{6}{2} = 15 \quad \text{membrane windings}$$

$$\binom{6}{5} = 6 \qquad \text{5-brane windings}$$

$P_M = (p_m, z^{mn}, z^{mnpqr})$ behave as momenta in a 27-dimensional space, and transform under $E_6(\mathbb{Z})$ U-duality. .

“Scalar fields” parametrise $E_{6(6)}/(USp(8)/\mathbb{Z}_2)$.
Parametrised by internal metric, 3-form and 6-form.

The discrete duality group contains the geometric mapping class group. Can it be “geometrised”?

Yes. This type of model goes under the name of **extended geometry**. Special cases are:
double geometry (for T-duality), and
exceptional geometry (for U-duality)

Early work by Tseytlin, Siegel, Hitchin, Hull, Zwiebach, …

A completely general framework has been formulated.

MC, Palmkvist, 1711.07694
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The gauge transformations in extended geometry
– the *generalised diffeomorphisms* –
unify diffeomorphisms and gauge transformations
for tensor fields.

$$\underbrace{L_U V^m = U^n\partial_n V^m}_{\text{transport term}} - \underbrace{\partial_n U^m V^n}_{\mathfrak{gl}\text{ transformation}}$$

$$\begin{aligned}\mathscr{L}_U V^M &= L_U V^M + Y^{MN}{}_{PQ}\partial_N U^P V^Q \\ &= \underbrace{U^N\partial_N V^M}_{\text{transport term}} + \underbrace{Z^{MN}{}_{PQ}\partial_N U^P V^Q}_{\mathfrak{g}\oplus\mathbb{R}\text{ transformation}}\end{aligned}$$

The transformations form an "algebra"

$$[\mathscr{L}_U, \mathscr{L}_V]W = \mathscr{L}_{[\![U,V]\!]}W$$

where the "Courant bracket" is $[\![U, V]\!] = \frac{1}{2}(\mathscr{L}_U V - \mathscr{L}_V U)$.
provided that the derivatives fulfil a "*section condition*".

The *section condition* ensures that fields locally depend only on an n-dimensional subspace of the coordinates, on which a $GL(n)$ subgroup acts. It reads $Y^{MN}{}_{PQ}\partial_M \ldots \partial_N = 0$.

The Courant bracket does not define a Lie algebra.

Given a KM algebra $\mathfrak{g}$ and a lowest weight coordinate representation $R(-\lambda)$, one may give a universal expression for the invariant tensors:

$$\sigma Y = -\eta^{\alpha\beta} T_\alpha \otimes T_\beta + (\lambda, \lambda) - 1 + \sigma$$

This is connected to the condition that any pair of momenta lie in a linear subspace of a minimal orbit of $R(\lambda)$.

Bossard, MC, Kleinschmidt, Samtleben, Palmkvist, 1708.08936
MC, Palmkvist, 1711.07694

Actually, the closure of the generalised diffeomorphisms happens only under certain conditions:

– The algebra $\mathfrak{g}$ is finite-dimensional, and

– $(\lambda, \theta) = 1$, *i.e.*, the highest weight λ is a fundamental weight, corresponding to a simple root with Coxeter label 1.

MC, Palmkvist, 1711.07694

In other cases, so-called ancillary transformations occur. These are restricted local $\mathfrak{g}$-tranformations, that remove degrees of freedom corresponding to “mixed tensors” (dual graviton, etc.)

Ancillary transformations are constructed with derivatives, and are restricted by the section constraint.

$$[\mathscr{L}_U, \mathscr{L}_V]W = \mathscr{L}_{\frac{1}{2}(\mathscr{L}_U V - \mathscr{L}_V U)}W + \Sigma_{U,V} \cdot W$$

The complete list of situations without ancillary transformations is:

- $\mathfrak{g} = A_r$, $\lambda = \Lambda_p$, $p = 1, \ldots, r$ (p-form representations);
- $\mathfrak{g} = B_r$, $\lambda = \Lambda_1$ (the vector representation);
- $\mathfrak{g} = C_r$, $\lambda = \Lambda_r$ (the symplectic-traceless r-form representation);
- $\mathfrak{g} = D_r$, $\lambda = \Lambda_1, \Lambda_{r-1}, \Lambda_r$ (the vector and spinor representations);
- $\mathfrak{g} = E_6$, $\lambda = \Lambda_1, \Lambda_5$ (the fundamental representations);
- $\mathfrak{g} = E_7$, $\lambda = \Lambda_1$ (the fundamental representation).

Gauge parameters: $U, V, \ldots \in R(-\lambda) = R_1$

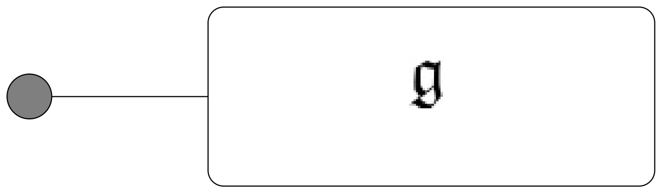
The Jacobi identity does not hold, but

$$[\![U[\![V,W]\!]]\!] + \text{cycl.} = d[\![U,V,W]\!]$$

where $[\![U,V,W]\!] \in R_2$, representing reducibility.

$R_1,\ R_2, \ldots$ are the positive levels (level = ghost number) of the Borcherds superalgebra $\mathscr{B}(\mathfrak{g})$ with Dynkin diagram

Berman, MC, Kleinschmidt, Thompson, 1208.5884
Palmkvist, 1507.08828
MC, Palmkvist, 1503.06215



This is the beginning of the L_∞ structure.

What is an L_∞ algebra?

Consider a full set of ghosts, including ghosts for ghosts, etc. Let $C = C_1 + C_2 + C_3 + \dots$
This is, to first approximation, an element in $\mathscr{B}_+(\mathfrak{g})$.

The Batalin–Vilkovisky action, restricted to ghosts, can be expanded as

$$S(C, C^\star) = \sum_{n=1}^{\infty} \langle C^\star, [\![C^n]\!]\rangle$$

where

$$[\![C^n]\!] = [\![\underbrace{C, C, \dots, C}_{n}]\!]$$

is the *n*-bracket.

The BV variation of C is

$$(S, C) = \sum_{n=1}^{\infty} [\![C^n]\!] \,.$$

In order for it to be nilpotent, $(S,(S,C)) = 0$, the brackets must satisfy the "generalised Jacobi identities"

$$\sum_{i=0}^{n-1} (i+1)[\![C^i, [\![C^{n-i}]\!]]\!] = 0$$

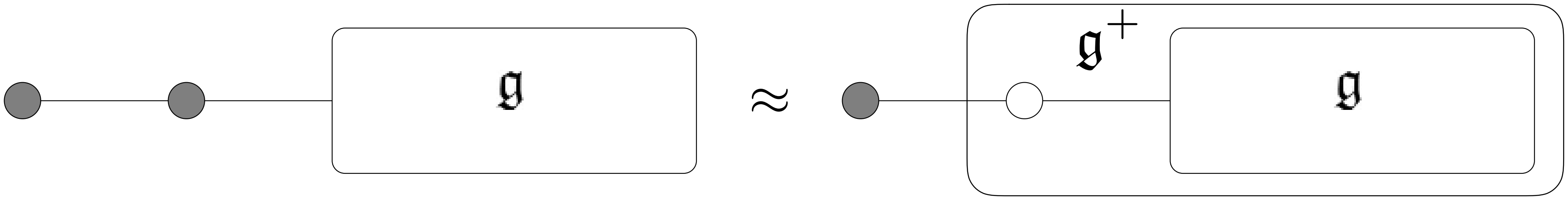
The derivative is the 1-bracket.

$$R_1 \xleftarrow[d]{} R_2 \xleftarrow[d]{} R_3 \xleftarrow[d]{} \dots$$

In order to be able to construct all the brackets, one needs to rely on some "underlying" structure. It would seems like the superalgebra $\mathscr{B}(\mathfrak{g})$ provides it.

However, there is no natural way of expressing the generalised Lie derivatives in terms of the (anti-)commutators of $\mathscr{B}(\mathfrak{g})$.

For this one needs a further extension, $\mathscr{B}(\mathfrak{g}^+)$:



Palmkvist, 1507.08828
MC, Palmkvist, 1711.07694

This is also good for several other reasons.

Using this Borcherds superalgebra as an underlying structure, we are able to construct all brackets as "derived brackets" and check their identities.

MC, Palmkvist, 1804.04377

Some interesting things appear in most cases, including the exceptional series. At some level (ghost number) the derivative fails to be a derivation, and the generalised Lie derivative fails to be covariant. This gives rise to extra ghosts, so-called ancillary ghosts. They are naturally encoded in the doubly extended algebra. (Also related to failure of Poincaré lemma.)

$$\begin{tikzcd}
 & & & & K_{p_0} \arrow[d, "\flat"'] & K_{p_0+1} \arrow[l, "d"] \arrow[d, "\flat"'] & K_{p_0+2} \arrow[l, "d"] \arrow[d, "\flat"'] & \cdots \arrow[l, "d"] \\
0 & C_1 \arrow[l, "d"] & \cdots \arrow[l, "d"] & C_{p_0-1} \arrow[l, "d"] & C_{p_0} \arrow[l, "d"] & C_{p_0+1} \arrow[l, "d"] & C_{p_0+2} \arrow[l, "d"] & \cdots \arrow[l, "d"]
\end{tikzcd}$$

Example: E_5

	$p=-1$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
$q=2$						$\mathbf{1}$	$\mathbf{16}$
$q=1$		$\mathbf{1}$	$\mathbf{16}$	$\mathbf{10}$	$\overline{\mathbf{16}}$	$\mathbf{45}\oplus\mathbf{1}$	$\overline{\mathbf{144}}\oplus\mathbf{16}$
$q=0$	$\overline{\mathbf{16}}$	$\mathbf{1}\oplus\mathbf{45}\oplus\mathbf{1}$	$\mathbf{16}$	$\mathbf{10}$	$\overline{\mathbf{16}}$	$\mathbf{45}$	$\overline{\mathbf{144}}$
$q=-1$	$\overline{\mathbf{16}}$	$\mathbf{1}$					

E_6 is on the diagonal $p = q$.

The marked fields constitute a so called tensor hierarchy. The modules are the ones of p-forms in the external dimension.

Concrete expression for all brackets:

$$\llbracket C \rrbracket = dC \ ,$$

$$\llbracket K \rrbracket = dK + K^\flat \ ,$$

$$\llbracket C^n \rrbracket = k_n \Big((\mathrm{ad}\, C)^{n-2} (\mathscr{L}_C C + X_C C) + \sum_{i=0}^{n-3} (\mathrm{ad}\, C)^i R_C (\mathrm{ad}\, C)^{n-i-3} \mathscr{L}_C C \Big)$$

$$\llbracket C^{n-1}, K \rrbracket = \frac{k_n}{n} \Big((\mathrm{ad}\, C)^{n-2} \mathscr{L}_C K + \sum_{i=0}^{n-3} (\mathrm{ad}\, C)^i \mathrm{ad}\, K (\mathrm{ad}\, C)^{n-i-3} \mathscr{L}_C C \Big) \ ,$$

where

$$k_{n+1} = \frac{2^n B_n^+}{n!} \ , \quad n \geqslant 1 \ .$$

The Borcherds superalgebra is unable to handle situations where ancillary transformations appear in the commutator of two generalised Lie derivatives. Then one needs a ***tensor hierarchy algebra***, a generalisation by Palmkvist of the Cartan-type superalgebras $W(n)$ and $S(n)$ in Kac's classification.

Palmkvist, 1305.0018
Carbone, MC, Palmkvist, 1802.05767
MC, Palmkvist, 1908.08695

These algebras agree with the Borcherds superalgebras at positive levels, and turn out to "know" when ancillary transformations appear. This development is necessary in order to go to higher dimensions (E_8, E_9,…),

Hohm, Samtleben, 1406.3348
MC, Rosabal, 1504.04843
Bossard, MC, Kleinschmidt, Samtleben, Palmkvist, 1708.08936
MC, Palmkvist, 1908.08696

but also for incorporating dynamical fields (vielbein, torsion,…) in the present framework.

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Torsion (the module of the embedding tensor of gauged supergravity) is contained at level -1, torsion Bianchi identities at level -2 etc.

Example: E_8

	$p=-1$	$p=0$	$p=1$	$p=2$
$q=2$			$\mathbf{1}$	$\mathbf{248}$
$q=1$	$\mathbf{1}\oplus\mathbf{3875}$	$\mathbf{248}$	$\mathbf{248}\oplus\mathbf{1}$	$\mathbf{1}\oplus\mathbf{3875}\oplus\mathbf{248}$
$q=0$	$\mathbf{248}\oplus\mathbf{1}\oplus\mathbf{3875}$	$\mathbf{1}\oplus\mathbf{248}$	$\mathbf{248}$	$\mathbf{1}\oplus\mathbf{3875}$
$q=-1$	$\mathbf{248}$	$\mathbf{1}$		

(ghosts)

Action formed using structure constants of *S(E₉)*.

Note the peculiar appearance of the bosonic subalgebra $\langle L_1\rangle \ltimes E_9$ at $p=q$. *Similar* phenomena become more pronounced for infinite-dimensional $\mathfrak{g}$.

Future:

Infinite-dimensional structure groups…

algebraic emergence of space (and time?)?

Extended supergeometry

Relaxation of section constraint?

…

Thank you!