Leptogenesis in a spatially flat Milne-type universe.

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Abstract

The quantum electrodynamics on a spatially flat (1+3)-dimensional Friedmann-Lematre-Robertson-Walker space-time with a Milne-type scale factor is considered focusing on the amplitudes of the allowed effects in the first order of perturbations. The definition of the transition rates is reconsidered obtaining an appropriate angular behavior of the probability of the pair creation from a photon which has a similar rate as the leptons creation from vacuum.

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Contents

Introduction

• In general relativity, the standard quantum field theory (QFT) based on perturbations is inchoate since one payed more attention to alternative non-perturbative methods as, for example, the cosmological particle creation [1, 2, 3, 4, 5, 6, 7, 8].

• The manifolds of actual interest in the actual cosmology are the spatially flat FLRW manifolds which are symmetric under translations and, consequently, there are quantum modes expressed in terms of plane waves with similar properties as in special relativity.

• These manifolds are useful for studying the behavior of the quantum matter in the presence of classical gravity turning back to the perturbation methods of the quantum field theory where significant results were obtained by many authors [9, 10, 11, 12, 13, 14, 15, 16].

• Inspired by these results we built the QED in Coulomb gauge on the de Sitter expanding universe [17], analyzing the processes in the first order of perturbations that are allowed on this manifold since the energy and momentum cannot be conserved simultaneously [9, 10, 11, 17].

• Recently we completed this approach with the integral representation of the fermion propagators we need for calculating Feynman diagrams in any order of perturbations [18]. Thus we have an example of a complete QED on the de Ssitter background.

• Looking for another example of manifold where the QED could be constructed without huge difficulties we observed that there exists an expanding space-time where the free field equations can be analytically solved [19].

This is the $(1+3)$-dimensional spatially flat FRLW manifold whose expansion is given by a Milne-type scale factor, proportional with the proper (or cosmic) time, t.

Milne's and Milne-type universes

The general metric in spherical coordinates of the $(1{+}3)$-dimensional FLRW manifolds,

$$ds^2 = dt^2 - a(t)^2 \left[\frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right] \tag{1}$$

with a Milne-type scale factor, $a(t) = \omega t$, depending on parameter ω, is produced by the sources

$$\rho = \frac{3}{8\pi G}\frac{\omega^2+\kappa}{\omega^2 t^2}\,, \quad p = -\frac{1}{8\pi G}\frac{\omega^2+\kappa}{\omega^2 t^2}\,, \tag{2}$$

Genuine Milne universe: $\omega = 1$ and $\kappa = -1 \to \ \rho = p = 0$

Spatially flat Milne-type univese (M): $\kappa = 0$ and arbitrary ω such that

$$\rho = \frac{3}{8\pi G}\frac{1}{t^2}\,, \quad p = -\frac{1}{8\pi G}\frac{1}{t^2}\,. \tag{3}$$

In this manifold we define the usual FLRW chart whose coordinates x^μ (labeled by the natural indices $\mu, \nu, ... = 0, 1, 2, 3$) are the proper time $x^0 = t$ and the Cartesian space coordinates, x^i ($i, j, k... = 1, 2, 3$), for which we may use the vector notation $\vec{x} = (x^1, x^2, x^3)$.

This chart, denoted by $\{t, \vec{x}\}$, is related to the conformal flat one, $\{t_c, \vec{x}\}$, where we have the same space coordinates but the conformal time $t_c \in (-\infty, \infty)$ defined as

$$t_c = \int \frac{dt}{a(t)} = \frac{1}{\omega} \ln(\omega t) \ \rightarrow \ a(t_c) = e^{\omega t_c} \,. \tag{4}$$

The corresponding line elements read

$$\begin{aligned} ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu &= dt^2 - (\omega t)^2 d\vec{x} \cdot d\vec{x} \\ &= e^{2\omega t_c}(dt_c^2 - d\vec{x} \cdot d\vec{x}) \,. \end{aligned} \tag{5}$$

The expansion of M that can be better observed in the chart $\{t, \vec{\hat{x}}\}$, of 'physical' space coordinates $\hat{x}^i = \omega t x^i$, where the line element

$$ds^2 = \left(1 - \frac{1}{t^2}\vec{\hat{x}} \cdot \vec{\hat{x}}\right) dt^2 + 2\vec{\hat{x}} \cdot d\vec{\hat{x}}\,\frac{dt}{t} - d\vec{\hat{x}} \cdot d\vec{\hat{x}}\,, \tag{6}$$

lays out an expanding horizon at $|\vec{\hat{x}}| = t$ and tends to the Minkowski space-time when $t \to \infty$ and the gravitational sources vanish.

In M we introduce the local orthogonal non-holonomic frames defined by the vector fields $e_{\hat{\alpha}} = e^{\mu}_{\hat{\alpha}}\partial_{\mu}$ and the associated co-frames given by the 1-forms $\omega^{\hat{\alpha}} = \hat{e}^{\hat{\alpha}}_{\mu} dx^{\mu}$, labeled by the local indices, $\hat{\mu}, \hat{\nu}, ... = 0, 1, 2, 3$. Here we use exclusively the diagonal tertrad gauge which preserves the symmetry of M as a global one,

$$e_0 = \partial_t = e^{-\omega t_c}\,\partial_{t_c}\,, \qquad \omega^0 = dt = e^{\omega t_c} dt_c\,, \tag{7}$$

$$e_i = \frac{1}{\omega t}\,\partial_i = e^{-\omega t_c}\,\partial_i\,, \qquad \omega^i = \omega t dx^i = e^{\omega t_c} dx^i\,. \tag{8}$$

Free fields on M

The massive Dirac field ψ of mass m which satisfy the field equation $(E_D - m)\psi = 0$ where

$$E_D = i\gamma^0\partial_t + i\frac{1}{\omega t}\gamma^i\partial_i + \frac{3i}{2}\frac{1}{t}\gamma^0 - m\,. \tag{9}$$

The term of this operator depending on the Hubble function $\frac{\dot{a}}{a} = \frac{1}{t}$ can be removed at any time by substituting $\psi \to (\omega t)^{-\frac{3}{2}}\psi$.

The fundamental solutions of the Dirac equation can be derived in the chiral representation (with diagonal γ^5) where we have to look for solutions of the form

$$U_{\vec{p},\sigma}(t,\vec{x}) = [2\pi a(t)]^{-\frac{3}{2}} e^{i\vec{p}\cdot\vec{x}}\mathcal{U}_p(t)u_\sigma \tag{10}$$

$$V_{\vec{p},\sigma}(t,\vec{x}) = [2\pi a(t)]^{-\frac{3}{2}} e^{-i\vec{p}\cdot\vec{x}}\mathcal{V}_p(t)v_\sigma \tag{11}$$

depending on the diagonal matrix-functions

$$\mathcal{U}_p(t) = \mathrm{diag}\left(u_p^+(t), u_p^-(t)\right) , \tag{12}$$

$$\mathcal{V}_p(t) = \mathrm{diag}\left(v_p^+(t), v_p^-(t)\right) , \tag{13}$$

whose matrix elements are functions only on t and $p = |\vec{p}|$, determining the time modulation of the fundamental spinors.

The spin part is encapsulated in the spinors of the momentum-helicity basis that in the chiral representation of the Dirac matrices read [29]

$$u_\sigma = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_\sigma(\vec{p}) \\ \xi_\sigma(\vec{p}) \end{pmatrix} \quad v_\sigma = \frac{c}{\sqrt{2}} \begin{pmatrix} -\eta_\sigma(\vec{p}) \\ \eta_\sigma(\vec{p}) \end{pmatrix} \tag{14}$$

where $\xi_\sigma(\vec{p})$ and $\eta_\sigma(\vec{p}) = i\sigma_2 \xi_\sigma^*$ are the Pauli spinors of the helicity basis corresponding to the helicities $\sigma = \pm\frac{1}{2}$ as given in the Appendix A.

The fundamental spinors are solutions of the free Dirac equation whether the modulation functions $u_p^{\pm}(t)$ and $v_p^{\pm}(t)$ satisfy the first order differential equations

$$\left(i\partial_t \pm \frac{2\sigma p}{\omega t}\right) u_p^{\pm}(t) \,=\, m\, u_p^{\mp}(t)\,, \tag{15}$$

$$\left(i\partial_t \mp \frac{2\sigma p}{\omega t}\right) v_p^{\pm}(t) \,=\, -m\, v_p^{\mp}(t)\,, \tag{16}$$

in the chart with the proper time. The solutions of these systems must satisfy the charge-conjugation symmetry [19],

$$v_p^{\pm}(t) = \left[u_p^{\mp}(t)\right]^* \,, \tag{17}$$

and the normalization conditions

$$|u_p^{+}|^2 + |u_p^{-}|^2 = |v_p^{+}|^2 + |v_p^{-}|^2 = 1\,. \tag{18}$$

that determine the definitive form of the fundamental spinors,

$$U_{\vec{p},\sigma}(x) = \sqrt{\frac{mt}{\pi}}\frac{e^{i\vec{p}\cdot\vec{x}}}{[2\pi\omega t]^{\frac{3}{2}}}\left(\begin{array}{c} K_{\sigma-i\frac{p}{\omega}}(im\,t)\,\xi_\sigma(\vec{p}) \\ K_{\sigma+i\frac{p}{\omega}}(im\,t)\,\xi_\sigma(\vec{p}) \end{array}\right) \tag{19}$$

$$V_{\vec{p},\sigma}(x) = \sqrt{\frac{mt}{\pi}}\frac{e^{-i\vec{p}\cdot\vec{x}}}{[2\pi\omega t]^{\frac{3}{2}}}\left(\begin{array}{c} K_{\sigma-i\frac{p}{\omega}}(-im\,t)\,\eta_\sigma(\vec{p}) \\ -K_{\sigma+i\frac{p}{\omega}}(-im\,t)\,\eta_\sigma(\vec{p}) \end{array}\right), \tag{20}$$

according to the identity (77).

The fundamental spinors (19) and (20) form the momentum-helicity basis in which the general solutions of the Dirac equation can be expanded as

$$\begin{aligned} \psi(t,\vec{x}\,) &= \psi^{(+)}(t,\vec{x}\,) + \psi^{(-)}(t,\vec{x}\,) \\ &= \int d^3p \sum_\sigma [U_{\vec{p},\sigma}(x)\mathfrak{a}(\vec{p},\sigma) + V_{\vec{p},\sigma}(x)\mathfrak{b}^\dagger(\vec{p},\sigma)]\,. \end{aligned} \tag{21}$$

After quantization, the particle $(\mathfrak{a}, \mathfrak{a}^\dagger)$ and antiparticle $(\mathfrak{b}, \mathfrak{b}^\dagger)$ operators satisfy the canonical anti-commutation relations [19],

$$\begin{aligned}\{\mathfrak{a}(\vec{p},\sigma), \mathfrak{a}^\dagger(\vec{p}\,', \sigma')\} &= \{\mathfrak{b}(\vec{p},\sigma), \mathfrak{b}^\dagger(\vec{p}\,', \sigma')\} \\ &= \delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}\,')\,.\end{aligned} \tag{22}$$

Then ψ becomes a quantum free field that can be used in perturbation for calculating physical effects.

The free Maxwell field A_μ can be written easily in the conformal chart taking over the well-known results in Minkowski space-time since the free Maxwell equations are conformally invariant. The electromagnetic gauge does not have this property such that we are forced to adopt the Coulomb gauge with $A_0(x) = 0$ as in Refs. [21, 17], remaining with the free Maxwell equations

$$\frac{1}{\sqrt{g(x)}}(\partial_{t_c}^2 - \Delta)A_i(x) = 0\,, \tag{23}$$

which can be solved in momentum-helicity basis where we obtain the expansion

$$A_i(x) = \int d^3k \sum_\lambda \left[\mu_{\vec{k},\lambda;\,i}(x)\alpha(\vec{k},\lambda) + \mu_{\vec{k},\lambda;\,i}(x)^*\alpha^\dagger(\vec{k},\lambda)\right] , \tag{24}$$

in terms of the modes functions,

$$\mu_{\vec{k},\lambda;\,i}(t_c,\vec{x}\,) = \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2k}}\, e^{-ikt_c + i\vec{k}\cdot\vec{x}}\,\varepsilon_i(\vec{k},\lambda)\,, \tag{25}$$

depending on the momentum $\vec{k}$ ($k = |\vec{k}|$) and helicity $\lambda = \pm 1$ of the polarization vectors $\vec{\varepsilon}_\lambda(\vec{k})$ in Coulomb gauge (given in Appendix A). Hereby we obtain the mode functions in the FLRW chart

$$\mu_{\vec{k},\lambda;\,i}(t,\vec{x}\,) = \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2k}}(\omega t)^{-i\frac{k}{\omega}}\, e^{i\vec{k}\cdot\vec{x}}\,\varepsilon_i(\vec{k},\lambda)\,. \tag{26}$$

First order QED amplitudes

The QED in Coulomb gauge on M can be constructed following step by step the method we used for the de Sitter QED [17]. The massive Dirac field ψ and the electromagnetic potential A_μ are minimally coupled to the gravity of M, interacting between themselves according to the QED action

$$\mathcal{S} = \int d^4x \sqrt{g}\, \left[\mathcal{L}_D(\psi) + \mathcal{L}_M(A) + \mathcal{L}_{\rm int}(\psi, A)\right] , \tag{27}$$

given by the Lagrangians of the Dirac (D) and Maxwell (M) free fields which have the standard form as in Ref. [17], while the interacting part,

$$\mathcal{L}_{\rm int}(\psi, A) = -e_0 \bar{\psi}(x)\gamma^{\hat{\mu}} e^{\nu}_{\hat{\mu}}(x) A_\nu(x)\psi(x)\,, \tag{28}$$

corresponds to the minimal electromagnetic coupling given by the electrical charge e_0.

The quantization of the entire theory and the perturbation procedure based on the reduction formalism can be done just as in the de Sitter case [17] exploiting usual $in-out$ initial/final conditions in the conformal chart where $t_c\in(-\infty,\infty)$. Finally, we obtain a perturbation procedure that allows us to calculate the transition amplitudes between two free states, $\alpha\to\beta$, that can be rewritten in the FLRW chart as

$$\langle out,\beta|in,\alpha\rangle=\langle\beta|Te^{\left(-i\int d^3x\sqrt{g}\int_0^\infty dt\mathcal{L}_{\rm int}\right)}|\alpha\rangle \tag{29}$$

where $\mathcal{L}_{\rm int}$ given by Eq. (28) is expressed in terms of free fields multiplied in the chronological order [27].

There are two types of processes involving particles, electrons of parameters $e^-(\vec{p},\sigma)$, antiparticles, $e^+(\vec{p}',\sigma')$ and photons $\gamma(\vec{k},\lambda)$.

1. The first type is when in and out states are charged as, for example, in the case of the photon adsorption $e^-+\gamma\to e^-$ whose amplitude reads

$$\begin{aligned}A_{\sigma'}^{\sigma;\lambda}(\vec{p},\vec{k};\vec{p}')&=\langle e^-(\vec{p}',\sigma')|\mathbf{S}_1|e^-(\vec{p},\sigma),\gamma(\vec{k},\lambda)\rangle\\&=-ie_0\int d^4x(\omega t)^2\,\overline{U}_{\vec{p}',\sigma'}(x)\,\gamma^i\mu_{\vec{k},\lambda;\,i}(x)U_{\vec{p},\sigma}(x)\,.\end{aligned} \tag{30}$$

When the photon is adsorbed by a positron we have to replace $U_{\vec{p}',\sigma'} \to V_{\vec{p},\sigma}$ and $U_{\vec{p},\sigma} \to V_{\vec{p}',\sigma'}$. Moreover, if we replace $\mu_i \to \mu_i^*$ then we obtain the amplitudes of the transitions $e^- \to e^- + \gamma$ and respectively $e^+ \to e^+ + \gamma$ in which a photon is emitted.

2. The second type of amplitudes involves only neutral in and out states as in the cases of the pair creation, $\gamma \to e^- + e^+$, and annihilation, $e^- + e^+ \to \gamma$, when we find the related amplitudes

$$\begin{aligned} A^{\lambda}_{\sigma,\sigma'}(\vec{k};\vec{p},\vec{p}') &= \langle e^-(\vec{p},\sigma), e^+(\vec{p}',\sigma')|\mathbf{S}_1|\gamma(\vec{k},\lambda)\rangle \\ &= -\langle \gamma(\vec{k},\lambda)|\mathbf{S}_1|e^-(\vec{p},\sigma), e^+(\vec{p}',\sigma')\rangle^* \\ &= -ie_0 \int d^4x(\omega t)^2\, \overline{U}_{\vec{p},\sigma}(x)\, \gamma^i \mu_{\vec{k},\lambda;\, i}(x) V_{\vec{p}',\sigma'}(x)\,. \end{aligned} \tag{31}$$

If we replace $\mu_i \to \mu_i^*$ in Eq. (31) then we obtain the amplitudes of the creation of leptons from vacuum, $vac \to e_+ + e_- + \gamma$ or their annihilation to vacuum, $e_+ + e_- + \gamma \to vac$.

In what follows we focus on the amplitudes (30) and (31) that can be calculated by using the previous results and taking into account that we work with the chiral representation of the Dirac matrices. Thus we obtain

$$A_{\sigma'}^{\sigma,\lambda}(\vec{p},\vec{k};\vec{p}\,') = i\frac{e_0 m}{\pi}\frac{\omega^{-i\frac{k}{\omega}-1}}{\sqrt{2k}\,(2\pi)^{\frac{3}{2}}}$$
$$\times\,\delta^3(\vec{p}+\vec{k}-\vec{p}\,')\,\Pi_{\sigma'}^{\sigma,\lambda}(\vec{p},\vec{k};\vec{p}\,')\,I^{-}_{\sigma',\sigma}(p',p,k)\,, \qquad (32)$$

$$A^{\lambda}_{\sigma,\sigma'}(\vec{k};\vec{p},\vec{p}\,') = i\frac{e_0 m}{\pi}\frac{\omega^{-i\frac{k}{\omega}-1}}{\sqrt{2k}\,(2\pi)^{\frac{3}{2}}}$$
$$\times\,\delta^3(\vec{p}+\vec{p}\,'-\vec{k})\,\Pi^{\lambda}_{\sigma,\sigma'}(\vec{k};\vec{p},\vec{p}\,')\,I^{+}_{\sigma,\sigma'}(p,p',k)\,, \qquad (33)$$

where we separate the terms depending on polarizations,

$$\Pi_{\sigma'}^{\sigma,\lambda}(\vec{p},\vec{k};\vec{p}\,') \;=\; \xi^{+}_{\sigma'}(\vec{p}\,')\sigma_i\varepsilon_i(\vec{k},\lambda)\xi_{\sigma}(\vec{p})\,, \qquad (34)$$
$$\Pi^{\lambda}_{\sigma,\sigma'}(\vec{k};\vec{p},\vec{p}\,') \;=\; \xi^{+}_{\sigma}(\vec{p})\sigma_i\varepsilon_i(\vec{k},\lambda)\eta_{\sigma'}(\vec{p}\,')\,, \qquad (35)$$

from the time integrals

$$I^{\pm}_{\sigma,\sigma'}(p,p',k) = \int_0^\infty dt\, \mathcal{K}^{\pm}_{\sigma,\sigma'}(p,p',k;t)\,, \tag{36}$$

whose time-dependent functions

$$\begin{aligned}\mathcal{K}^{\pm}_{\sigma,\sigma'}(p,p',k;t) = t^{i\frac{k}{\omega}} \Big[& K_{\sigma+i\frac{p}{\omega}}(-imt) K_{\sigma'-i\frac{p'}{\omega}}(\mp imt) \\ & \pm K_{\sigma-i\frac{p}{\omega}}(-imt) K_{\sigma'+i\frac{p'}{\omega}}(\mp imt) \Big]\,,\end{aligned} \tag{37}$$

result from Eqs. (19) and (20).

These integrals have remarkable properties,

$$\begin{aligned} I^{\pm}_{\sigma,\sigma'}(p,p',k) &= \pm I^{\pm}_{-\sigma,-\sigma'}(p,p',k) = \pm I^{\pm}_{\sigma,\sigma'}(-p,-p',k) \\ &= I^{\pm}_{\sigma,-\sigma'}(p,-p',k) = I^{\pm}_{-\sigma,\sigma'}(-p,p',k)\,, \end{aligned} \tag{38}$$

since $K_\nu = K_{-\nu}$, and can be solved according to Eq. (79) obtaining, after a few manipulations, the following quantities we need for deriving the transition probabilities:

$$\left| I^{+}_{\pm\frac{1}{2},\pm\frac{1}{2}}(p,p',k) \right| = \Delta(p,p',k)\, e^{\frac{\pi k}{2\omega}}\,, \tag{39}$$

$$\left| I^{+}_{\mp\frac{1}{2},\pm\frac{1}{2}}(p,p',k) \right| = \Delta(p,-p',k)\, e^{\frac{\pi k}{2\omega}}\,, \tag{40}$$

$$\begin{aligned} \left| I^{-}_{\pm\frac{1}{2},\pm\frac{1}{2}}(p,p',k) \right| &= \Delta(p,p',k)\, e^{\frac{\pi k}{2\omega}} \\ &\times \left| \sinh\frac{\pi p}{\omega} \pm \frac{p'-p}{k} \cosh\frac{\pi p}{\omega} \right|\,, \end{aligned} \tag{41}$$

$$\begin{aligned} \left| I^{-}_{\mp\frac{1}{2},\pm\frac{1}{2}}(p,p',k) \right| &= \Delta(p,-p',k)\, e^{\frac{\pi k}{2\omega}} \\ &\times \left| \sinh\frac{\pi p}{\omega} \pm \frac{p+p'}{k} \cosh\frac{\pi p}{\omega} \right|\,, \end{aligned} \tag{42}$$

where

$$\Delta(p,p',k) = \frac{\pi^{\frac{3}{2}}\sqrt{\omega}}{2m} \left[\frac{k\sinh\frac{k\pi}{\omega}}{k^2-(p-p')^2}\right]^{\frac{1}{2}}$$
$$\times \left[\sinh\left(\frac{\pi(k-p+p')}{2\omega}\right)\sinh\left(\frac{\pi(k+p-p')}{2\omega}\right)\right.$$
$$\left.\times \cosh\left(\frac{\pi(k+p+p')}{2\omega}\right)\cosh\left(\frac{\pi(k-p-p')}{2\omega}\right)\right]^{-\frac{1}{2}} . \tag{43}$$

We observe that the function $\Delta(p,p',k)$ satisfies

$$\Delta(p,p',k) = \Delta(-p,-p',k) = \Delta(p,p',-k)\,, \tag{44}$$

being singular for $k \pm (p-p') = 0$. Note that the function $\Delta(p,-p',k)$ is singular only for $k = (p+p')$ since $k,p,p' \in \mathbb{R}^+$.

Rates and probabilities

The transition amplitudes of processes $\alpha \to \beta$ have the general form

$$A_{\alpha\beta} = \langle out\, \beta | in\, \alpha \rangle = \delta^3(\vec{p}_\alpha - \vec{p}_\beta) M_{\alpha\beta} I_{\alpha\beta}\,, \tag{45}$$

laying out the Dirac δ-function of the momentum conservation but without conserving the energy. Thus the time integration gives the quantity

$$I_{\alpha\beta} = \int_0^\infty dt\, \mathcal{K}_{\alpha\beta}(t)\,, \tag{46}$$

instead of the familiar $\delta(E_\alpha - E_\beta)$ we meet in the flat case when the energy is conserved. This could lead to some difficulties when we calculate the transition probabilities.

We remind the reader that in the usual QED on Minkowski space-time the transition probabilities are derived from amplitudes satisfying the energy-momentum conservation,

$$\hat{A}_{\alpha\beta} = \delta(E_\alpha - E_\beta) \delta^3(\vec{p}_\alpha - \vec{p}_\beta) \hat{M}_{\alpha\beta}\,, \tag{47}$$

evaluating $\delta(0)\delta^3(0) \sim \frac{1}{(2\pi)^4}TV$ in terms of the total volume V and interaction time T such that one obtains the probability per unit of volume and unit of time as [28, 27]

$$\hat{\mathcal{P}}_{\alpha\beta} = \frac{|\hat{A}_{\alpha\beta}|^2}{VT} = \delta(E_\alpha - E_\beta)\delta^3(\vec{p}_\alpha - \vec{p}_\beta)\frac{|\hat{M}_{\alpha\beta}|^2}{(2\pi)^4}\,. \tag{48}$$

In fact this is the transition rate per unit of volume we refer here simply as rate denoted by $\mathcal{R}$.

In our QED on M the rates must be derived in another manner since the amplitudes have here different forms as in Eq. (45). Therefore, we introduce first the time-dependent amplitudes

$$\begin{aligned} A_{\alpha\beta}(t) &= \delta^3(\vec{p}_\alpha - \vec{p}_\beta)M_{\alpha\beta}I_{\alpha\beta}(t) \\ &= \delta^3(\vec{p}_\alpha - \vec{p}_\beta)M_{\alpha\beta}\int_0^t dt'\mathcal{K}_{\alpha\beta}(t')\,, \end{aligned} \tag{49}$$

that can be rewritten in terms of the conformal time as $A_{\alpha\beta}(t_c) = A_{\alpha\beta}[t(t_c)]$.

Then we define the transition rate according to Eq. (4) as

$$\mathcal{R}_{\alpha\beta} = \lim_{t_c\to\infty} \frac{1}{2V}\frac{d}{dt_c}\left|A_{\alpha\beta}(t_c)\right|^2 = \lim_{t\to\infty} \frac{\omega t}{2V}\frac{d}{dt}\left|A_{\alpha\beta}(t)\right|^2 \tag{50}$$

obtaining the final result

$$\mathcal{R}_{\alpha\beta} = \delta^3(\vec{p}_\alpha - \vec{p}_\beta)\frac{|M_{\alpha\beta}|^2}{(2\pi)^3}|I_{\alpha\beta}|K_{\alpha\beta} \tag{51}$$

where

$$K_{\alpha\beta} = \lim_{t\to\infty}\left|\omega t\,\mathcal{K}_{\alpha\beta}(t)\right| . \tag{52}$$

Note that the basic definition (50) is given in the conformal chart where the in and out states can be defined correctly in the domain $-\infty < t_c < \infty$, as in the flat case or in our de Sitter QED.

Thus for calculating the transition rates of the processes under consideration here we need to calculate the limits (52) of the functions (37). Fortunately, this can be done easily since the modified Bessel functions have a simple asymptotic behavior as in Eq. (78). Thus we obtain the dramatic result,

$$\lim_{t\to\infty} \omega t \left| \mathcal{K}^{+}_{\sigma,\sigma'}(p,p',k;t) \right| = \frac{\pi\omega}{m}\,, \tag{53}$$

$$\lim_{t\to\infty} \omega t \left| \mathcal{K}^{-}_{\sigma,\sigma'}(p,p',k;t) \right| = 0\,, \tag{54}$$

which shows that the rates of all the processes involving charged states vanish, remaining only with the transitions between neutral states.

Moreover, we observe that in the flat limit, for $\omega \to 0$, all the transition rates vanishes in the first order of perturbations as was expected since in special relativity these processes are forbiden by the energy-momentum conservation [27].

Now we focus on the remaining transition, $\gamma(\vec{k},\lambda) \to e^{-}(\vec{p},\sigma) + e^{+}(\vec{p}\,',\sigma')$ for which we obtain the rate

$$\mathcal{R}^{\lambda}_{\sigma,\sigma'}(\vec{k};\vec{p},\vec{p}') = \frac{e_0^2}{(2\pi)^7}\,\frac{m\omega}{k}\,\delta^3(\vec{p}+\vec{p}'-\vec{k})$$
$$\times\,|\Pi^{\lambda}_{\sigma,\sigma'}(\vec{k};\vec{p},\vec{p}')|^2\,|I^{+}_{\sigma,\sigma'}(p,p',k)|\,, \tag{55}$$

which allows us to derive the probability per units of volume and time integrating over $\vec{k}$. Thus we obtain

$$\mathcal{P}^{\lambda}_{\sigma,\sigma'}(\vec{p},\vec{p}') = \int \frac{d^3k}{(2\pi)^3}\,\mathcal{R}^{\lambda}_{\sigma,\sigma'}(\vec{k};\vec{p},\vec{p}') = \frac{e_0^2}{(2\pi)^{10}}\,\frac{m\omega}{k(\theta)}$$
$$\times\,|\Pi^{\lambda}_{\sigma,\sigma'}(\vec{p}+\vec{p}';\vec{p},\vec{p}')|^2\,|I^{+}_{\sigma,\sigma'}(p,p',k(\theta))|\,, \tag{56}$$

where

$$k(\theta) = \left|\vec{p}+\vec{p}'\right| = \sqrt{p^2+2pp'\cos\theta+p'^2} \tag{57}$$

depend on the angle θ between $\vec{p}$ and $\vec{p}'$.

For studying these probabilities we need to calculate the polarization terms which are extremely complicated in an arbitrary geometry. Therefore, we

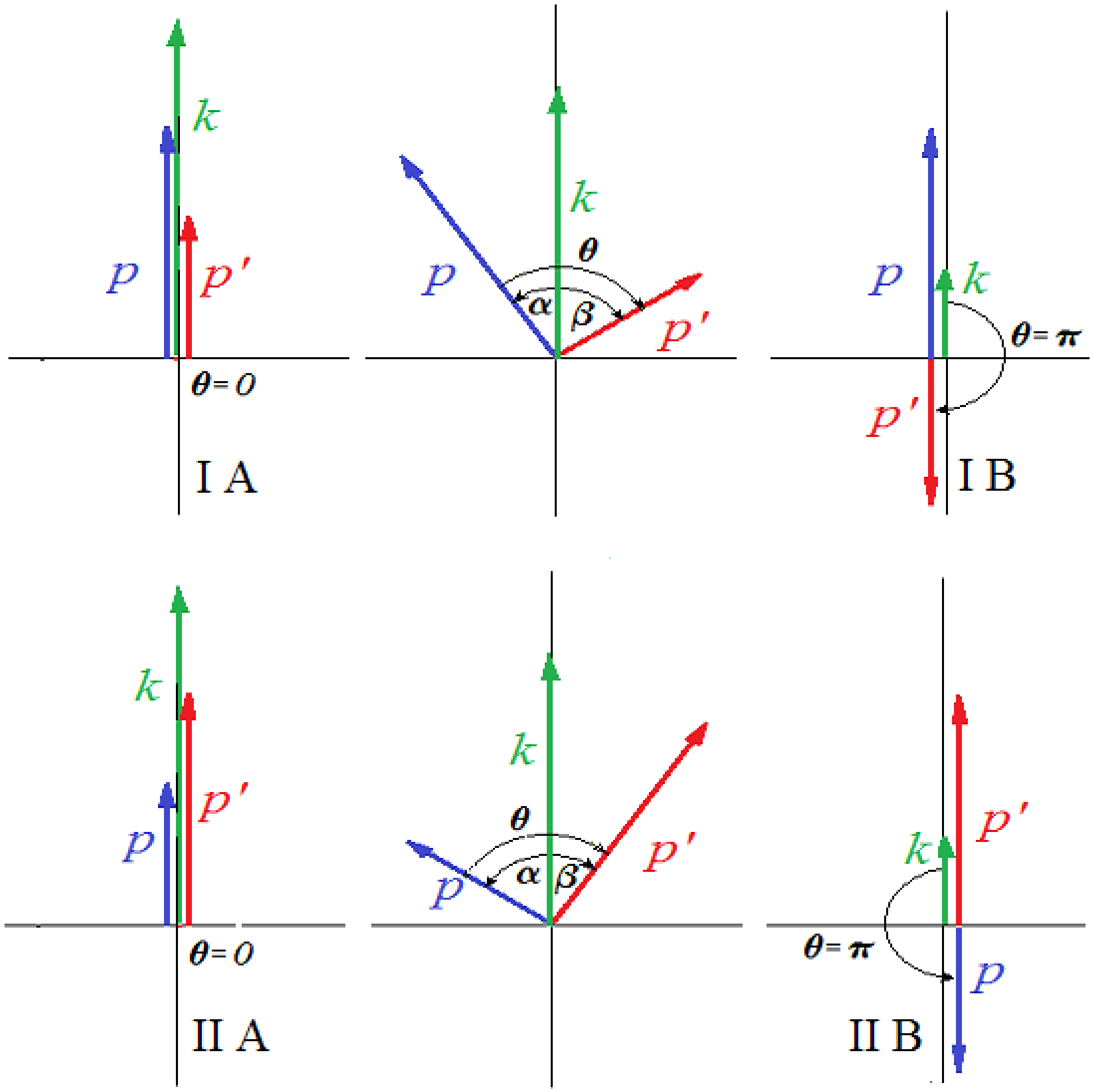

Figure 1: Pair production in the frame $\{e\}$ (I) for $p > p'$: (I A) $\theta = 0 \rightarrow k = p + p'$, $\sigma' = \sigma$ and $\lambda = 2\sigma$, (I B) $\theta = \pi \rightarrow k = p' - p$, $\sigma' = -\sigma$ and $\lambda = 2\sigma$, and (II) for $p < p'$: (II A) $\theta = 0 \rightarrow k = p + p'$, $\sigma' = \sigma$ and $\lambda = 2\sigma$, (II B) $\theta = \pi \rightarrow k = p' - p$ and $\sigma' = -\sigma$ and $\lambda = -2\sigma$.

consider a particular frame $\{e\} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ in the momentum space where $\vec{k} = \vec{p} + \vec{p}\,' = k(\theta)\vec{e}_3$ and the vectors $\vec{p}$ and $\vec{p}\,'$ are in the plane $\{\vec{e}_1, \vec{e}_3\}$ (as in Fig. 1) having the spherical coordinates $\vec{p} = (p, \alpha, 0)$ and $\vec{p}\,' = (p', \beta, \pi)$ such that

$$\theta = \alpha + \beta\,, \tag{58}$$

$$p \sin\alpha = p' \sin\beta\,. \tag{59}$$

In this geometry the polarization vectors take the simple form $\vec{\varepsilon}_{\pm 1}(\vec{k}) = \frac{1}{\sqrt{2}}(\pm\vec{e}_1 - i\vec{e}_2)$ that allows us to derive the polarization matrices

$$\hat{\Pi}^{\lambda=1} = \sqrt{2}\, \begin{pmatrix} \cos\frac{\alpha}{2}\cos\frac{\beta}{2} & \cos\frac{\alpha}{2}\sin\frac{\beta}{2} \\ \sin\frac{\alpha}{2}\cos\frac{\beta}{2} & \sin\frac{\alpha}{2}\sin\frac{\beta}{2} \end{pmatrix}\,, \tag{60}$$

$$\hat{\Pi}^{\lambda=-1} = \sqrt{2}\, \begin{pmatrix} \sin\frac{\alpha}{2}\sin\frac{\beta}{2} & \sin\frac{\alpha}{2}\cos\frac{\beta}{2} \\ \cos\frac{\alpha}{2}\sin\frac{\beta}{2} & \cos\frac{\alpha}{2}\cos\frac{\beta}{2} \end{pmatrix}\,, \tag{61}$$

whose matrix elements, $\left|\hat{\Pi}^{\lambda}_{\sigma,\sigma'}\right|$ are the absolute values of the polarization terms in the particular frame $\{e\}$. Now we can choose the free parameters

p, p' and θ since the angles we need for calculating the polarization matrix can be deduced as

$$\alpha = \arctan\left(\frac{p' \sin\theta}{p + p' \cos\theta}\right) , \tag{62}$$

$$\beta = \theta - \arctan\left(\frac{p' \sin\theta}{p + p' \cos\theta}\right) , \tag{63}$$

when $p > p'$, as it results from Eqs. (58) and (59). For $p < p'$ we obtain similar relations changing $\alpha \leftrightarrow \beta$ and $p \leftrightarrow p'$ while for $p = p'$ we have $\alpha = \beta = \frac{\theta}{2}$. Then, according to Eqs. (39) and (40) we obtain the definitive result in the frame $\{e\}$ where the probability per unit of volume and unit of time,

$$\begin{aligned} \mathcal{P}^{\lambda}_{\sigma,\sigma'}(p, p', \theta) &= \frac{e_0^2}{(2\pi)^{10}} \frac{m\omega}{k(\theta)} e^{\frac{\pi k(\theta)}{2\omega}} \\ &\times \hat{\Pi}^2_{\sigma,\sigma'} \Delta(p, \mathrm{sign}(\sigma\sigma')p', k(\theta)) , \end{aligned} \tag{64}$$

depends only on polarization and the free parameters (p, p', θ) through the polarization term and the function $\Delta(p, p', k)$ defined by Eq. (43).

A similar result can be obtained for the process of lepton creation, $vac \to \gamma + e^- + e^+$, with similar parameters, whose rates or probabilities comply with the general rule

$$\frac{\mathcal{P}_{vac \to \gamma + e^- + e^+}(p, p', \theta)}{\mathcal{P}_{\gamma \to e^- + e^+}(p, p', \theta)} \simeq e^{-\frac{\pi k(\theta)}{\omega}} . \tag{65}$$

Thus we remain only with the processes of pair creation and lepton creation or with the combined leptonic creation $vac \to \gamma + e^- + e^+ \to (e^- + e^+) + e^- + e^+$ since the transitions between charged states are forbidden.

Note that the inverse processes of pair annihilation, $e^- + e^+ \to \gamma$, or lepton annihilation to vacuum, $\gamma + e^- + e^+ \to vac$ cannot be produced since it is less probable that two or three particles meet each other spontaneously.

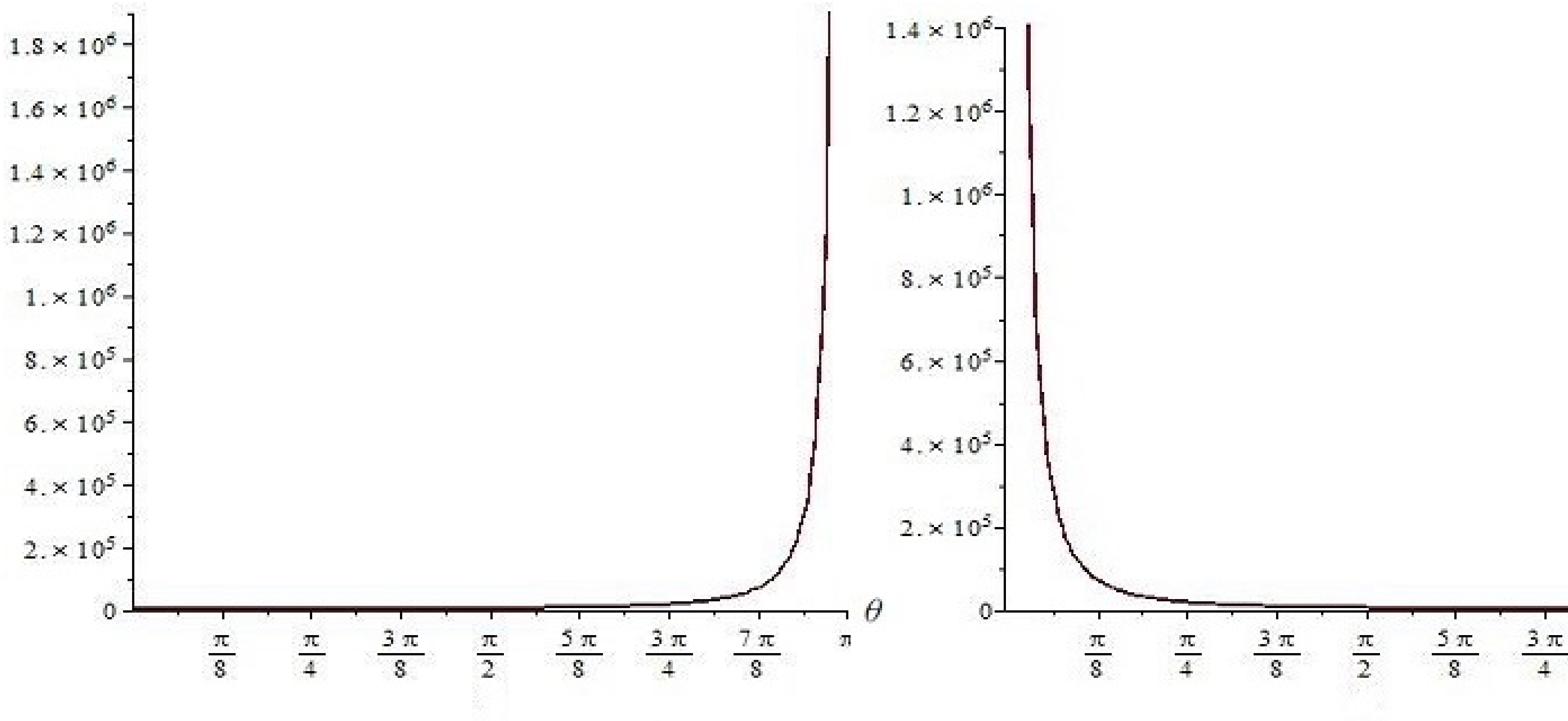

Figure 2: The singular behavior of the functions $\Delta(p, p', k(\theta))$ (left panel) and $\Delta(p, -p', k(\theta))$ (right panel) for $p = 0.01\,\omega$ and $p' = 0.03\,\omega$.

Graphical analysis

We observe first that here we cannot speak about the polarization conservation since we work in the momentum-helicity basis. Nevertheless, there are some particular positions in which the momenta have the same direction and, consequently, the polarizations must be conserved as spin projections on the same direction. These positions are obtained either for $\theta = 0$, as in the panels I A and II A of Fig. 1, when we have

$$\alpha = \beta = 0 \to p' = p + k \,, \quad \lambda = \sigma + \sigma' \,, \tag{66}$$

or for $\theta = \pi$ when we find two different cases presented in the panels II A and respectively II B. In the first one (I B) we set $p > p'$ and consequently

$$\alpha = 0 \,, \beta = \pi \to k = p - p' \,, \quad \lambda = \sigma - \sigma' \,, \tag{67}$$

while in the second one (II B) the situation is reversed such that $p < p'$ and

$$\alpha = \pi \,, \beta = 0 \to k = p' - p \,, \quad \lambda = \sigma' - \sigma \,. \tag{68}$$

Note that when $p = p'$ we remain only with the parallel case (I A=II A) since the anti-parallel equal momenta lead to $k = 0$ when the photon of the in state disappears.

Now we expect to recognize the above selection rules by plotting the probabilities (64) versus θ for fixed values of the momenta p and p'. The unpleasant surprise is of finding a wrong behaviors just for the angles $\theta = 0$ or $\theta = \pi$ for which the selection rules require the probabilities to vanish if the polarizations are not conserved. This is because of the function $\Delta(p, p', k(\theta))$ which becomes singular for $k \pm (p - p') = 0$ having the profile plotted in Fig. 2.

Thus we meet again the sickness of the perturbation procedures leading to singularities or violation of the conservation rules on some particular directions.

In order to extract the physical information we need to remove these effects resorting to the method of Yennie et al. [22] of constructing the reduced

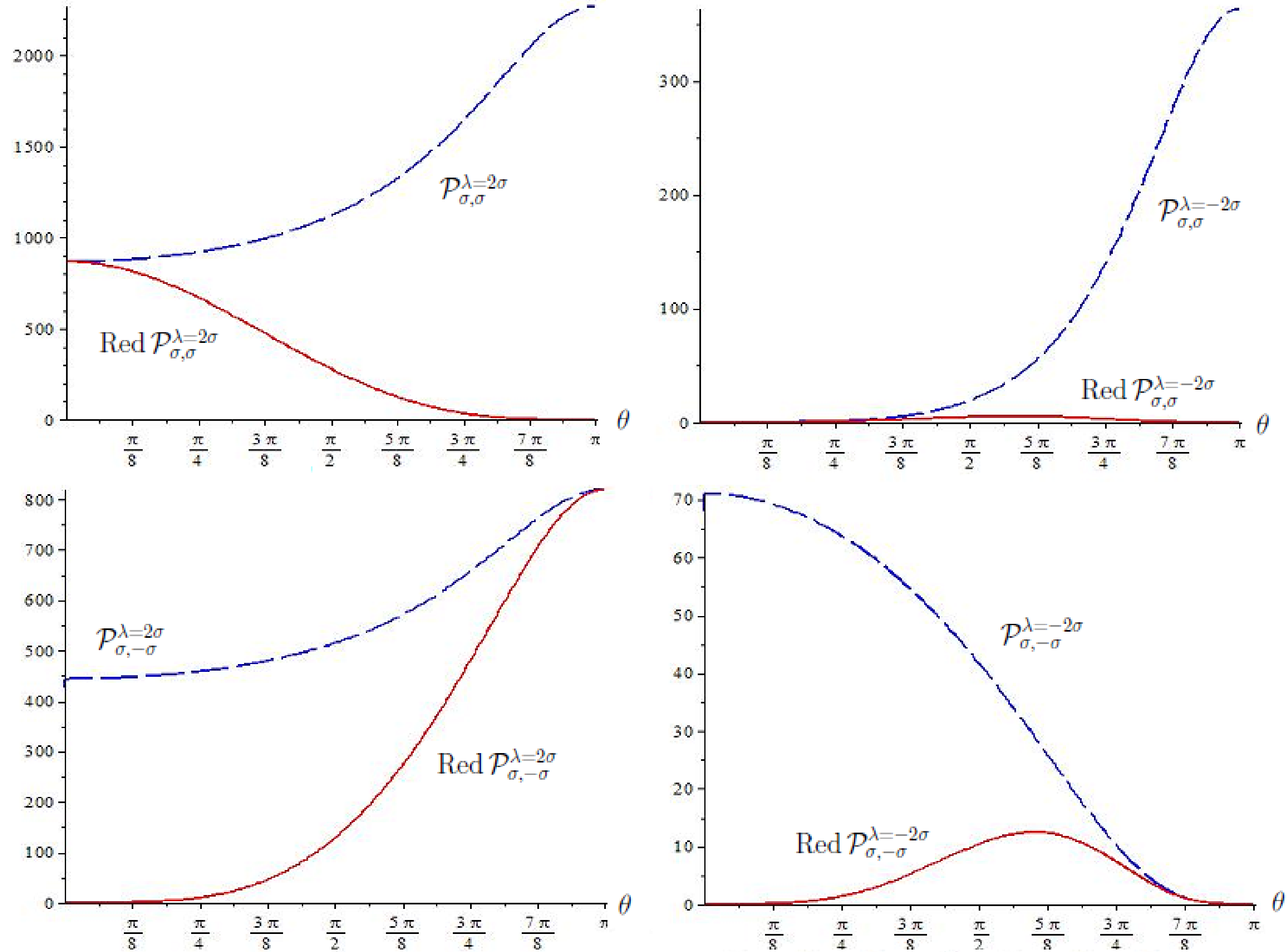

Figure 3: The effect of the reduction procedure: the original (dashed lines) and reduced (solid lines) probabilities versus θ for different polarizations and $p = 0.05\,\omega$ and $p' = 0.02\,\omega$.

amplitudes by multiplying the calculated one by suitable trigonometric functions. Thus, for example, the singularity at $\theta = 0$ of the scattering amplitudes of various scattering processes can be removed by multiplying the amplitude with $(1-\cos\theta)^n$ where n gives the reduction order.

In the case of our amplitudes the reduction of the first order, with $n = 1$, is enough for eliminating the singularities in $\theta = 0$ and $\theta = \pi$ if we define the reduced probabilities as

$$\mathrm{Red}\,\mathcal{P}^{\lambda=\pm 2\sigma}_{\sigma,\sigma}(p,p',\theta) = \mathcal{P}^{\lambda=\pm 2\sigma}_{\sigma,\sigma}(p,p',\theta)\cos^4\frac{\theta}{2}\,, \tag{69}$$

$$\mathrm{Red}\,\mathcal{P}^{\lambda=\pm 2\sigma}_{\sigma,-\sigma}(p,p',\theta) = \mathcal{P}^{\lambda=\pm 2\sigma}_{\sigma,-\sigma}(p,p',\theta)\sin^4\frac{\theta}{2}\,. \tag{70}$$

Now we can verify that these match perfectly with the selection rules (66)-(68) by plotting them on the whole domain $\theta \in [0,\pi]$ as in Figs. 3 and 4. Moreover, we observe that the reduction procedure does not affect the physical content since for the angles $\theta = 0$ and $\theta = \pi$ for which the function Δ is regular we have $\mathrm{Red}\,\mathcal{P}^{\lambda}_{\sigma,\sigma'} = \mathcal{P}^{\lambda}_{\sigma,\sigma'}$ as we see in Fig. 3. Thus we can conclude that the reduction procedure is correct helping us to understand the physical behavior of the analyzed process.

On the other hand, we must specify that another problem is the divergence at $p \sim p' = 0$. Indeed, as we see in Fig. 4, the reduced probabilities increase when the momenta p and p' are decreasing such that for vanishing momenta the probabilities diverge,

$$\lim_{\substack{p\to 0\\ p'\to 0}} \mathcal{P}^{\lambda}_{\sigma,\sigma'} = \lim_{\substack{p\to 0\\ p'\to 0}} \mathrm{Red}\,\mathcal{P}^{\lambda}_{\sigma,\sigma'} = \infty\,. \tag{71}$$

This unwanted effect is somewhat analogous to the infrared catastrophe of the usual QED and could be of interest in a future procedure of the vertex renormalization.

Finally we note that the dependence on the parameter ω is almost trivial since for large values of ω the probabilities are increasing linearly with this parameter.

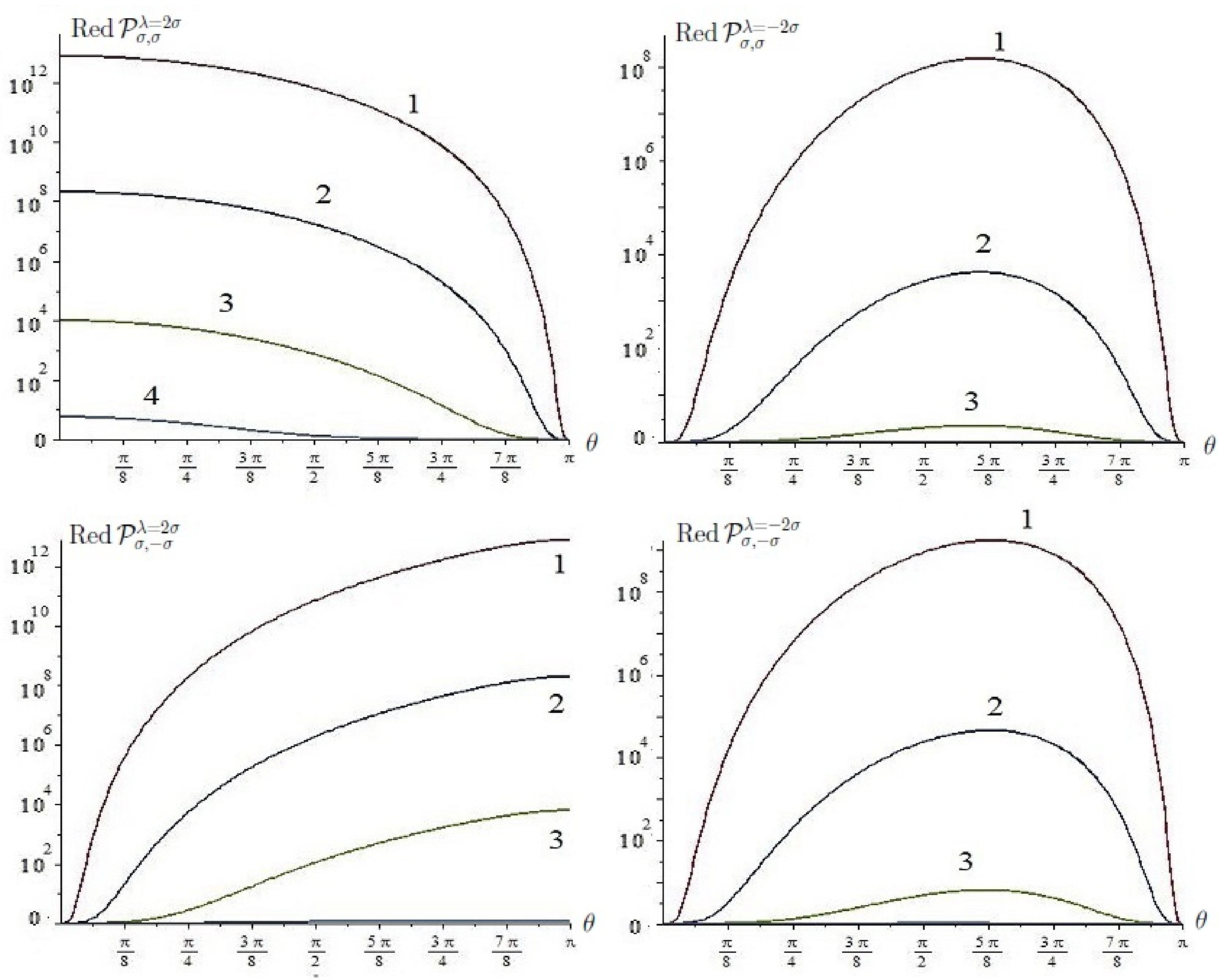

Figure 4: Reduced probabilities versus θ for different polarizations and momenta: (1) $p = 0.002\,\omega$ and $p' = 0.001\,\omega$ (2) $p = 0.02\,\omega$ and $p' = 0.01\,\omega$. (3) $p = 0.2\,\omega$ and $p' = 0.1\,\omega$ (4) $p = 2\,\omega$ and $p' = \omega$

Concluding remarks

We visited here for the first time the world of the quantum fields on the spatially flat FLRW space-time with a Milne-type modulation factor (denoted here by M).

The first impression was that this manifold, born from a time singularity, might produce new spectacular physical effects but, in fact, our calculations show that, at least from the point of view of the quantum theory, this space-time behaves normally producing similar effects as the de Sitter expanding universe [17].

The only notable new feature is that the first order transitions between charged states are forbidden but we cannot say if this is specific to this geometry as long as we do not have other examples.

From the technical point of view, M and the de Sitter space-time have complementary behaviors as we can see from the next self-explanatory table,

$$
\begin{array}{lcc}
 & M & \text{de Sitter} \\
t & 0 < t = \frac{1}{\omega}e^{\omega t_c} < \infty & -\infty < t < \infty \\
t_c & -\infty < t_c < \infty & -\infty < t_c = -\frac{1}{\omega}e^{-\omega t} < -\frac{1}{\omega} \\
a(t) & \omega t & e^{\omega t} \\
a(t_c) & e^{\omega t_c} & -\frac{1}{\omega t_c} \\
u_p^{\pm} & K_{\frac{1}{2}\mp i\frac{2\sigma p}{\omega}}(imt) & K_{\frac{1}{2}\mp i\frac{m}{\omega}}(ipt_c)
\end{array}
$$

where we denote by ω the free parameter of M and the Hubble constant of the de Sitter expanding portion [26]. Thus we have at least two related examples that will help us to construct the perturbative QFT on curved backgrounds.

APPENDIX A: Polarization

The Pauli spinors of the momentum-helicity basis, $\xi_\sigma(\vec{p})$, of helicity $\sigma = \pm\frac{1}{2}$, satisfy the eigenvalues problem $(\vec{p}\cdot\vec{S})\,\xi_\sigma(\vec{p}) = \sigma\, p\, \xi_\sigma(\vec{p})$ where $S_i = \frac{1}{2}\sigma_i$ are the spin operators expressed in terms of Pauli matrices. They have the form

$$\xi_{\frac{1}{2}}(\vec{p}) = \sqrt{\frac{p+p^3}{2p}}\left(\begin{array}{c} 1 \\ \frac{p^1+ip^2}{p+p^3} \end{array}\right), \tag{72}$$

$$\xi_{-\frac{1}{2}}(\vec{p}) = \sqrt{\frac{p+p^3}{2p}}\left(\begin{array}{c} \frac{-p^1+ip^2}{p+p^3} \\ 1 \end{array}\right). \tag{73}$$

The antiparticle spinors are defined usually as $\eta_\sigma(\vec{p}) = i\sigma_2\xi_\sigma(\vec{p})^*$ [27, 29] in order to satisfy $(\vec{p}\cdot\vec{S})\,\eta_\sigma(\vec{p}) = -\sigma\, p\, \eta_\sigma(\vec{p})$.

The polarization of the free Maxwell field is given by the polarization vectors $\vec{\varepsilon}_\lambda(\vec{k})$ which have c-number components. Here we consider only the *circular* polarization [27] with $\vec{\varepsilon}_{\pm 1}(\vec{k}) = \frac{1}{\sqrt{2}}(\pm\vec{e}_1 + i\vec{e}_2)$, in a three-dimensional orthogonal local frame $\{\vec{e}_i\}$ where $\vec{k} = k\vec{e}_3$.

APPENDIX B: Modified Bessel functions

According to the general properties of the modified Bessel functions, $I_\nu(z)$ and $K_\nu(z) = K_{-\nu}(z)$ [30], we deduce that those used here, $K_{\nu_\pm}(z)$, with $\nu_\pm = \frac{1}{2} \pm i\mu$ are related among themselves through

$$H_\nu^{(1,2)}(z) = \mp \frac{2i}{\pi} e^{\mp \frac{i}{2}\pi\nu} K_\nu(\mp iz)\,, \quad z \in \mathbb{R}\,. \tag{74}$$

The functions used here, $K_{\nu_\pm}(z)$ with $\nu_\pm = \frac{1}{2} \pm i\mu$ ($\mu \in \mathbb{R}$), are related among themselves through

$$[K_{\nu_\pm}(z)]^* = K_{\nu_\mp}(z^*)\,, \quad \forall z \in \mathbb{C}\,, \tag{75}$$

satisfy the equations

$$\left(\frac{d}{dz} + \frac{\nu_\pm}{z}\right) K_{\nu_\pm}(z) = -K_{\nu_\mp}(z)\,, \tag{76}$$

and the identities

$$K_{\nu_\pm}(iz)K_{\nu_\mp}(-iz) + K_{\nu_\pm}(-iz)K_{\nu_\mp}(iz) = \frac{\pi}{|z|}\,, \tag{77}$$

that guarantees the correct orthonormalization properties of the fundamental spinors. For $z \to \infty$ these functions behave as [30]

$$I_\nu(z) \to \sqrt{\frac{\pi}{2z}} e^z\,, \quad K_\nu(z) \to K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}\,, \tag{78}$$

regardless the index ν.

Moreover, here we use the integral (6576-4) of Ref. [31] with $b = \pm a$,

$$\int_0^\infty dx\, x^{-\lambda} K_\mu(ax) K_\nu(\pm ax) = \frac{(\pm)^\nu\, 2^{-2-\lambda} a^{\lambda-1}}{\Gamma(1-\lambda)} \times \Gamma\left(\frac{1-\lambda+\mu+\nu}{2}\right) \Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right) \times \Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right) \Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right) . \quad (79)$$

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