Entropy in Poincaré gauge theory: Hamiltonian approach

Milutin Blagojević i Branislav Cvetković[1]

[1] Institute of physics, Belgrade

УНИВЕРЗИТЕТ У БЕОГРАДУ
ИНСТИТУТ ЗА ФИЗИКУ | БЕОГРАД
ИНСТИТУТ ОД НАЦИОНАЛНОГ
ЗНАЧАЈА ЗА РЕПУБЛИКУ СРБИЈУ

10th MATHEMATICAL PHYSICS MEETING
Belgrade, 12.09.2019.

Talk overview

The talk is based on the following papers

- M. Blagojević and B. Cvetković, Entropy in Poincaré gauge theory: Hamiltonian approach, Phys. Rev. D **99**, 104058 (2019)
- M. Blagojević and B. Cvetković, Hamiltonian approach to black hole entropy: Kerr-like spacetimes, Phys. Rev. D **100**, 044029 (2019)

- Already in 1960s, Kibble and Sciama proposed a new theory of gravity, the Poincaré gauge theory (PG), based on gauging the Poincaré group of spacetime symmetries.
- PG is characterized by a Riemann-Cartan (RC) *geometry* of spacetime, in which both the torsion and the curvature are essential ingredients of the *gravitational dynamics*.
- Nowadays, PG is a well-established approach to gravity, representing a natural gauge-field-theoretic extension of general relativity (GR).
- In the past half century, many investigations of PG have been aimed at clarifying different aspects of both the geometric and dynamical roles of torsion. In particular, successes in constructing exact solutions with torsion naturally raised the question of how their *conserved charges* are influenced by the presence of torsion.

- We shall reconsider the notion of conserved charge in the Hamiltonian formalism, as it represents the most natural basis for the main subject of the present talk, the *influence of torsion* on black hole entropy.
- The expressions for the conserved charges in PG were first found for asymptotically flat solutions. In the Hamiltonian approach to PG the conserved charges are represented by a boundary term, defined by requiring the variation of the canonical gauge generator to be a well-defined (differentiable) functional on the phase space.
- A covariant version of the Hamiltonian approach, introduced later by Nester, turned out to be an important step in understanding the conservation laws. This was clearly demonstrated by Hecht and Nester, in their analysis of the conserved charges for asymptotically flat or (A)dS.

- Despite an intensive activity in exploring the notion of conserved charges in the *generic* four-dimensional (4D) PG, systematic studies of black hole entropy in the presence of torsion have been largely neglected so far.
- One should mention here an early and general proposal by Nester which did not prove to be quite successful.
- Later investigations were restricted to EC theory, which is certainly not sufficient to justify any conclusion on the general relation between torsion and entropy.
- In 3D gravity, black hole entropy is well understood for solutions possessing the asymptotic conformal symmetry.
- The physics of black holes is an arena where thermodynamics, gravity, and quantum theory are connected through the existence of entropy as an intrinsic dynamical aspect of black holes.

- In the 1990s, understanding of the *classical* black hole entropy reached a level that can be best characterized by Wald's words: "Black hole entropy is the Noether charge" .
- The question that we wish to address is whether such a challenging idea can improve our understanding of black hole entropy in PG.
- We constructed the canonical gauge generator in the first order formulation of PG, which improved form is used to obtain the variational equation for the asymptotic canonical charge, located at the spatial 2-boundary at infinity.
- Following the idea that "entropy is the canonical charge at horizon," we are led to define black hole entropy by the same variational equation, located at black hole horizon.
- The differentiability of the gauge generator guarantees the validity of the first law of black hole thermodynamics.

- Our conventions are as follows.
- The greek indices $(\mu, \nu, \dots)$ refer to the coordinate frame, with a time-space splitting expressed by $\mu = (0, \alpha)$.
- The latin indices $(i, j, \dots)$ refer to the local Lorentz frame.
- b^i is the orthonormal tetrad (1-form), h_i is the dual basis (frame), with $h_i \rfloor b^k = \delta_i^k$, and the Lorentz metric is $\eta_{ij} = (1, -1, -1, -1)$.
- The volume 4-form is $\hat{\epsilon} = b^0 b^1 b^2 b^3$, the Hodge dual of a form α is $^\star\alpha$, with $^\star 1 = \hat{\epsilon}$, and the totally antisymmetric tensor is defined by $^\star(b_i b_j b_m b_n) = \varepsilon_{ijmn}$, where $\varepsilon_{0123} = +1$.
- The exterior product of forms is implicit.

- Basic dynamical variables of PG are the tetrad field b^i and the spin connection ω^{ij} (1-forms), the gauge potentials related to the translation and the Lorentz subgroups of the Poincaré group, respectively. The corresponding field strengths are the torsion $T^i = db^i + \omega^i{}_m b^m$ and the curvature $R^{ij} = d\omega^{ij} + \omega^i{}_m \omega^{mj}$ (2-forms).
- Varying the gravitational Lagrangian $L_G = L_G(b^i, T^i, R^{ij})$ (4-form) with respect to b^i and ω^{ij} yields the gravitational field equations *in vacuum*. After introducing the covariant field momenta, $H_i := \partial L_G/\partial T^i$ and $H_{ij} := \partial L_G/\partial R^{ij}$, and the associated energy-momentum and spin currents, $E_i := \partial L_G/\partial b^i$ and $E_{ij} := \partial L_G/\partial \omega^{ij}$, the equations read

$$\delta b^i : \qquad \nabla H_i + E_i = 0\,, \tag{2.1a}$$

$$\delta \omega^{ij} : \qquad \nabla H_{ij} + E_{ij} = 0\,. \tag{2.1b}$$

- Assuming the gravitational Lagrangian L_G to be at most quadratic in the field strengths and parity invariant,

$$L_G = -{}^\star(a_0 R + 2\Lambda) + T^i \sum_{n=1}^{3} {}^\star(a_n {}^{(n)}T_i) + \frac{1}{2} R^{ij} \sum_{n=1}^{6} {}^\star(b_n {}^{(n)}R_{ij}) \,,$$

the gravitational field momenta take the form

$$H_i = 2\sum_{m=1}^{3} {}^\star(a_m {}^{(m)}T_i)\,, \qquad H_{ij} = -2a_0 {}^\star(b^i b^j) + H'_{ij}\,,$$

$$H'_{ij} := 2\sum_{n=1}^{6} {}^\star(b_n {}^{(n)}R_{ij})\,.$$

- Here, (a_0, a_m, b_n) are the Lagrangian parameters, with $16\pi a_0 = 1$, Λ is a cosmological constant, and ${}^{(m)}T_i$ and ${}^{(n)}R_{ij}$ are irreducible parts of torsion and curvature.

- A black hole can be described as a region of spacetime which is causally disconnected from the rest of spacetime.
- The boundary of a black hole is a null hypersurface, known as the *event horizon*.
- Let us consider a black hole characterized by the existence of a Killing vector field ξ. A null hypersurface to which the Killing vector is normal, is called the *Killing horizon* ($\mathcal{K}$). As a consequence, $\xi^2 := g_{\mu\nu}\xi^\mu\xi^\nu = 0$ on $\mathcal{K}$. The gradient $\partial_\mu(\xi^2)$ is also normal to $\mathcal{K}$ and it must be proportional to ξ_μ,

$$\partial_\mu(\xi^2) = -2\kappa\xi_\mu\,, \qquad (2.2)$$

  where the scalar function κ is known as *surface gravity*.
- One can show, without making use of any field equations, that for a wide class of stationary black holes (systems in “equilibrium"), the Killing horizon coincides with event horizon.

- The essential property of surface gravity is expressed by the *zeroth law* of black hole mechanics: For a wide class of stationary black holes, surface gravity is constant over the entire event horizon.
- Since null geodesics and Killing vector fields are purely metric notions, they can be directly transferred to PG. Thus, the form of surface gravity and the associated zeroth law of black mechanics are also valid *in PG.*
- The calculation of κ should be done in coordinates that are well defined on the outer horizon, such as ingoing Edington-Finkelstein coordinates, where the metric reads

$$ds^2 = N^2 dv^2 - 2dv\,dr - r^2 d\Omega^2\,, \qquad N = N(r)\,, \tag{2.3}$$

while the definition (2.2) of surface gravity takes the form

$$\partial_r N^2 = 2\kappa\,. \tag{2.4}$$

- In PG, the conserved charges are determined as the values of the (improved) canonical generators of spacetime symmetries, associated to suitable asymptotic conditions.
- The canonical procedure is simplified by transforming the quadratic Lagrangian into the “first order" form

$$L_G = T^i \tau_i + \frac{1}{2} R^{ij} \rho_{ij} - V(b^i, \tau_i, \rho_{ij}) \,, \qquad (3.1)$$

  where the gravitational potentials (b^i, ω^{ij}) and “covariant momenta" (τ_i, ρ_{ij}), are *independent* dynamical variables.
- The potential V is a quadratic function of (τ_i, ρ_{ij}) which ensures the on-shell relations $\tau_i = H_i$ and $\rho_{ij} = H_{ij}$.
- In the tensor formalism, the Lagrangian density reads

$$\tilde{\mathcal{L}}_G = -\frac{1}{4} \varepsilon^{\mu\nu\lambda\rho} \left(T^i{}_{\mu\nu} \tau_{i\lambda\rho} + \frac{1}{2} R^{ij}{}_{\mu\nu} \rho_{ij\lambda\rho} \right) - \tilde{\mathcal{V}}(b, \tau, \rho) \,. \qquad (3.2)$$

- The gravitational field equations (in vacuum) are obtained by varying $\tilde{\mathcal{L}}_G$ with respect to the independent dynamical variables $b^i{}_\mu, \omega^{ij}{}_\mu, \tau^i{}_{\mu\nu}$ and $\rho^{ij}{}_{\mu\nu}$:

$$\nabla_\mu{}^{(\star)}\tau_i{}^{\mu\nu} - \frac{\partial\tilde{\mathcal{V}}}{\partial b^i{}_\nu} = 0\,, \tag{3.3a}$$

$$2b_{[j\mu}{}^{(\star)}\tau_{i]}{}^{\mu\nu} + \nabla_\mu \rho_{ij}{}^{\mu\nu} = 0\,, \tag{3.3b}$$

$$-{}^{(\star)}T^{i\mu\nu} - \frac{\partial\tilde{\mathcal{V}}}{\partial \tau_{i\mu\nu}} = 0\,, \tag{3.3c}$$

$$-{}^{(\star)}R^{ij\mu\nu} - \frac{\partial\tilde{\mathcal{V}}}{\partial \rho_{ij\mu\nu}} = 0\,, \tag{3.3d}$$

where we use the notation ${}^{(\star)}\tau_i{}^{\mu\nu} := \frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}\tau_{i\lambda\rho}$, and similarly for ${}^{(\star)}\rho_{ij}{}^{\mu\nu}$, ${}^{(\star)}T^{i\mu\nu}$ and ${}^{(\star)}R^{ij\mu\nu}$.

- Starting with the field variables $\varphi^A = (b^i{}_\mu, \omega^{ij}{}_\mu, \tau^i{}_{\mu\nu}, \rho^{ij}{}_{\mu\nu})$ and the corresponding canonical momenta $\pi_A = (\pi_i{}^\mu, \pi_{ij}{}^\mu, P_i{}^{\mu\nu}, P_{ij}{}^{\mu\nu})$, one obtains the following primary constraints:

$$
\begin{aligned}
&\phi_i{}^0 := \pi_i{}^0 \approx 0\,, &&\phi_i{}^\alpha := \pi_i{}^\alpha + {}^{(\star)}\tau_i{}^{0\alpha} \approx 0\,,\\
&\phi_{ij}{}^0 := \pi_{ij}{}^0 \approx 0\,, &&\phi_{ij}{}^\alpha := \pi_{ij}{}^\alpha + \frac{1}{2}{}^{(\star)}\rho_{ij}{}^{0\alpha} \approx 0\,,\\
&P_i{}^{\mu\nu} \approx 0\,, &&P_{ij}{}^{\mu\nu} \approx 0\,.
\end{aligned}
\qquad (3.4)
$$

The canonical Hamiltonian is found to have the form

$$
\begin{aligned}
H_c &:= b^i{}_0\mathcal{H}_i + \frac{1}{2}\omega^{ij}{}_0\mathcal{H}_{ij} + \tau_{i0\alpha}{}^{(\star)}T^{i0\alpha} + \frac{1}{2}\rho_{ij0\alpha}{}^{(\star)}R^{ij0\alpha} + \tilde{\mathcal{V}}\,,\\
\mathcal{H}_i &:= \nabla_\alpha{}^{(\star)}\tau_i{}^{0\alpha}\,,\\
\mathcal{H}_{ij} &:= 2b_{[j\alpha}{}^{(\star)}\tau_{i]}{}^{0\alpha} + \nabla_\alpha{}^{(\star)}\rho_{ij}{}^{0\alpha}\,.
\end{aligned}
\qquad (3.5)
$$

- The total Hamiltonian reads

$$H_T := H_c + u^i{}_\mu \phi_i{}^\mu + \frac{1}{2} u^{ij}{}_\mu \phi_{ij}{}^\mu + \frac{1}{2} v^i{}_{\mu\nu} P_i{}^{\mu\nu} + \frac{1}{4} v^{ij}{}_{\mu\nu} P_{ij}{}^{\mu\nu} ,$$

  where u's and v's are canonical multipliers.
- The consistency conditions of the sure primary constraints produces the secondary constraints

$$\hat{\mathcal{H}}_i := \mathcal{H}_i + \frac{\partial \tilde{\mathcal{V}}}{\partial b^i{}_0} \approx 0 \, , \ \hat{\mathcal{H}}_{ij} := \mathcal{H}_{ij} \approx 0 \, ,$$

$$\hat{\mathcal{T}}^{i0\alpha} := {}^{(\star)}T^{i0\alpha} + \frac{\partial \tilde{\mathcal{V}}}{\partial \tau_{i0\alpha}} \approx 0 \, , \ \hat{\mathcal{R}}^{ij0\alpha} := {}^{(\star)}R^{ij0\alpha} + \frac{\partial \tilde{\mathcal{V}}}{\partial \rho_{ij0\alpha}} \approx 0 \, ,$$

  which correspond to certain components of the field equations (3.3).
- The remaining primary constraints are second class.

- We can construct the corresponding DB and use them in the consistency procedure on the reduced phase space:

$$\{b^i{}_\alpha, \tau_{j\beta\gamma}\}^* = \delta^i_j \varepsilon_{0\alpha\beta\gamma}\,, \qquad \{\omega^{ij}{}_\alpha, \rho_{kl\beta\gamma}\}^* = \delta^{[i}_k \delta^{j]}_l \varepsilon_{0\alpha\beta\gamma}\,.$$

- The form of the total Hamiltonian is simplified:

$$H_T = H_c + u^i{}_0 \pi_i{}^0 + \frac{1}{2} u^{ij}{}_0 \pi_{ij}{}^0 + v^i{}_{0\beta} P_i{}^{0\beta} + \frac{1}{2} v^{ij}{}_{0\beta} P_{ij}{}^{0\beta}\,. \quad (3.6)$$

- In terms of the secondary constraints H_c reads

$$H_c = b^i{}_0 \hat{\mathcal{H}}_i + \frac{1}{2}\omega^{ij}{}_0 \mathcal{H}_{ij} + \tau_{i0\alpha} \hat{\mathcal{T}}^{i0\alpha} + \frac{1}{2}\rho_{ij0\alpha} \hat{\mathcal{R}}^{ij0\alpha}. \quad (3.7)$$

- A phase-space functional G is a good gauge generator if it generates the correct gauge transformations of all phase-space variables.

- Relying on an explicit construction of G in 3D PG, we display here its generalization to 4D:

$$G[\xi,\theta]=\int_\Sigma d^3x\,(G_1+G_2)\,,\; G_2=\frac{1}{2}\dot\theta^{ij}\pi_{ij}{}^0+\frac{1}{2}\theta^{ij}\mathcal{M}_{ij}\,,$$

$$G_1=\dot\xi^\mu\left(b^i{}_\mu\pi_i{}^0+\frac{1}{2}\omega^{ij}{}_\mu\pi_{ij}{}^0+\tau^i{}_{\mu\beta}P_i{}^{0\beta}+\frac{1}{2}\rho^{ij}{}_{\mu\beta}P_{ij}{}^{0\beta}\right)+\xi^\mu\mathcal{P}_\mu$$

$$\mathcal{P}_\mu:=b^i{}_\mu\hat{\mathcal{H}}_i+\frac{1}{2}\omega^{ij}{}_\mu\mathcal{H}_{ij}+\tau^i{}_{\mu\beta}\hat{\mathcal{T}}_i{}^{0\beta}+\frac{1}{2}\rho^{ij}{}_{\mu\beta}\hat{\mathcal{R}}_{ij}{}^{0\beta}$$

$$+(\partial_\mu b^i{}_0)\pi_i{}^0+\frac{1}{2}(\partial_\mu\omega^{ij}{}_0)\pi_{ij}{}^0+(\partial_\mu\tau^i{}_{0\beta})P_i{}^{0\beta}+\frac{1}{2}(\partial_\mu\rho^{ij}{}_{0\beta})P_{ij}{}^{0\beta}$$

$$-\partial_\beta\left(\tau^i{}_{0\mu}P_i{}^{0\beta}+\frac{1}{2}\rho^{ij}{}_{0\mu}P_{ij}{}^{0\beta}\right)\,,$$

$$\mathcal{M}_{ij}:=\mathcal{H}_{ij}+2\left(b_{[i0}\pi_{j]}{}^0+\omega^k{}_{[i0}\pi_{kj]}{}^0+\tau_{[i0\gamma}P_{j]}{}^{0\gamma}+\rho^k{}_{[i0\gamma}P_{kj]}{}^{0\gamma}\right)$$

- The Hamiltonian formulation of gravity is based on the existence of a family of spacelike hypersurfaces Σ, labeled by the time parameter t. Each Σ is bounded by a closed 2-surface at spatial infinity, which is used to define the *asymptotic charge*. When Σ is a black hole manifold, it also possesses an “interior" boundary, the horizon, which serves to define *black hole entropy*.
- In PG, conserved charges are closely related to the canonical gauge generator G. Since G acts on dynamical variables via the PB (or DB) operation, it should have well-defined functional derivatives. If G does not satisfy this requirement the problem can be solved by adding a suitable surface term Γ_∞, located at the boundary of Σ at infinity, such that $\tilde{G} = G + \Gamma_\infty$ is well defined. The value of Γ_∞ is exactly the canonical charge of the system.

- Any particular solution of PG is characterized by a set of asymptotic conditions. Demanding that local Poincaré transformations preserve these conditions, one obtains certain restrictions on the Killing-Lorentz parameters. The restricted parameters define the asymptotic symmetry, which is essential for the existence of conserved charges.
- We consider the variation of the gauge generator

$$
\begin{aligned}
\delta G &= \int_\Sigma d^3x(\delta G_1 + \delta G_2)\,, \\
\delta G_1 &= \xi^\mu\Big[b^i{}_\mu\delta\hat{\mathcal{H}}_i + \frac{1}{2}\omega^{ij}{}_\mu\delta\mathcal{H}_{ij} + \tau_{i\mu\alpha}\delta\hat{\mathcal{T}}^{i0\alpha} + \frac{1}{2}\rho_{ij\mu\alpha}\delta\hat{\mathcal{R}}^{ij0\alpha}\Big]\,, \\
\delta G_2 &= \frac{1}{2}\theta^{ij}\delta\mathcal{H}_{ij} + R\,,
\end{aligned}
\tag{4.1}
$$

where δ is the variation over the set of asymptotic states, and R denotes regular (differentiable) terms.

- To get rid of the unwanted $\delta\partial_\mu\varphi$ terms which spoil the differentiability of G, one can perform a partial integration,

$$\delta G_1 = \frac{1}{2}\varepsilon^{0\alpha\beta\gamma}\partial_\alpha\Big\{\xi^\mu\Big[b^i{}_\mu\delta\tau_{i\beta\gamma} + \frac{1}{2}\omega^{ij}{}_\mu\delta\rho_{ij\beta\gamma} + 2\tau_{i\mu\gamma}\delta b^i{}_\beta$$

$$+\rho_{ij\mu\gamma}\delta\omega^{ij}{}_\beta\Big]\Big\} + R\,,\ \delta G_2 = \frac{1}{2}\varepsilon^{0\alpha\beta\gamma}\partial_\alpha\Big[\frac{1}{2}\theta^{ij}\delta\rho_{ij\beta\gamma}\Big]\,.$$

- Going over to the notation of differential forms we get

$$\delta G = -\delta\Gamma_\infty + R\,,\quad \delta\Gamma_\infty := \oint_{S_\infty}\delta B\,, \tag{4.2a}$$

$$\delta B := (\xi\rfloor b^i)\delta H_i + \delta b^i(\xi\rfloor H_i) + \frac{1}{2}(\xi\rfloor\omega^{ij})\delta H_{ij} + \frac{1}{2}\delta\omega^{ij}(\xi\rfloor H_{ij})$$

$$+\frac{1}{2}\theta^{ij}\delta H_{ij}\,, \tag{4.2b}$$

where S_∞ is the boundary of Σ at infinity.

- If the asymptotic conditions ensure Γ_∞ to be a *finite* solution of the variational equation (4.2), the improved gauge generator
$$\tilde{G} := G + \Gamma_\infty \qquad (4.3)$$
has well-defined functional derivatives. Then, the value of $\tilde{G}$ is effectively given by the value of Γ_∞, which represents the canonical *charge at infinity*.
  (a1) In the above variational equations, the variation of Γ_∞ is defined over a suitable set of asymptotic states, keeping the background configuration fixed.
- Nester and co-workers succeeded to explicitly construct a set of finite expressions Γ_∞. Although their approach yields highly reliable expressions for the conserved charges, we shall continue using the variational approach (4.2), as it can be naturally extended to a new definition of black hole entropy.

- In order to interpret black hole entropy as the canonical charge on horizon, we assume that the boundary of Σ has two components, one at spatial infinity and the other at horizon, $\partial\Sigma = S_\infty \cup S_H$.
- Now the condition of differentiability of the canonical generator G includes two boundary terms, the integrals of $\delta B = \delta B(\xi, \theta)$ over S_∞ and S_H:

$$\delta G = -\oint_{S_\infty} \delta B + \oint_{S_H} \delta B + R\,. \tag{4.4}$$

- Here, as we already know, the first term represents the asymptotic canonical charge,

$$\delta \Gamma_\infty = \int_{S_\infty} \delta B\,. \tag{4.5}$$

- The second one defines entropy S as the canonical *charge on horizon*,

$$\delta\Gamma_H := \oint_{S_H} \delta B\,. \tag{4.6}$$

(a2) The variation of Γ_H is performed by varying the parameters of a solution, but keeping surface gravity constant.

- Explicit form of entropy depends on two factors: dynamical and geometric properties of a theory and specific structure of the black hole.
- For stationary black holes in GR, the entropy formula (4.6) takes the well-known form

$$\delta\Gamma_H = T\delta S\,, \tag{4.7}$$

where $T = \kappa/2\pi$ represents the temperature and $S = \pi r_+^2$ is black hole entropy.

- The gauge generator G is regular if and only if the sum of two boundary terms vanishes,

$$\delta\Gamma_\infty - \delta\Gamma_H = 0\,, \tag{4.8}$$

which is nothing but the *first law* of black hole thermodynamics. Thus, the validity of the first law directly follows from the regularity of the original gauge generator G.
- In the framework of PG, the conserved charge is a well-established concept which has been calculated for a number of exact solutions. In contrast to that, much less is known about black hole entropy.
- We shall now test our definition of black hole entropy and the associated first law, on illustrative examples from the family of Schwarzschild-AdS solutions.

- All exact solutions of GR are also solutions of PG. However, certain properties of a solution may change when we go from GR to a new dynamical environment of PG.
- We shall first discuss the case of the Riemannian Schwarzschild-AdS black hole in PG, defined by the metric

$$ds^2 = N^2 dt^2 - \frac{dr^2}{N^2} - r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \qquad N^2 := 1 - \frac{2m}{r} + \lambda r^2\,,$$

and $\lambda > 0$. The zeros of N^2 determine the event horizon

$$\lambda r^3 + r - 2m = 0\,. \tag{5.1}$$

- This equation has just one real root r_+, which is positive iff $m > 0$, and N^2 is positive in the region $r > r_+$, where the Schwarzschild-like coordinates are well defined.

- The surface gravity and black hole temperature are

$$\kappa = \frac{1}{2r_+}(3\lambda r_+^2 + 1)\,, \qquad T = \frac{\kappa}{2\pi}\,. \qquad (5.2)$$

- The orthonormal tetrad is chosen in the form

$$b^0 = Ndt\,, \quad b^1 = \frac{dr}{N}\,, \quad b^2 = rd\theta\,, \quad b^3 = r\sin\theta d\varphi\,. \quad (5.3)$$

- The Riemannian connection reads

$$\omega^{01} = -N'b^0\,, \qquad \omega^{1c} = \frac{N}{r}b^c\,, \qquad \omega^{23} = \frac{\cos\theta}{r\sin\theta}b^3\,, \quad (5.4)$$

and the corresponding curvature 2-form R^{ij} is

$$^{(6)}R^{ij} = \lambda b^i b^j\,, \quad ^{(1)}R^{01} = -\frac{2m}{r^3}b^0 b^1\,, \quad ^{(1)}R^{23} = -\frac{2m}{r^3}b^2 b^3\,,$$

$$^{(1)}R^{Ac} = \frac{m}{r^3}b^A b^c\,, \quad A = (0,1)\,,\ c = (2,3)\,. \qquad (5.5)$$

- The covariant momenta are $H_i = 0$ and

$$
\begin{aligned}
H_{01} &= -2b^2 b^3 \left(a_0 + 2b_1 \frac{m}{r^3} - b_6 \lambda \right), \\
H_{23} &= -2b^0 b^1 \left(a_0 + 2b_1 \frac{m}{r^3} - b_6 \lambda \right), \\
H_{Ac} &= -\varepsilon_{Acmn} b^m b^n \left(a_0 - b_1 \frac{m}{r^3} - b_6 \lambda \right). \qquad (5.6)
\end{aligned}
$$

- One can show that the Riemannian Schwarzschild-AdS spacetime is an exact solution of PG, provided that

$$3a_0\lambda + \Lambda = 0\,. \qquad (5.7)$$

- Energy of the Riemannian Schwarzschild-AdS solution in PG can be calculated from the variational formula for $\xi = \partial_t$ and $\theta^{ij} = 0$. The result is

$$E = 16\pi A_0 m, \qquad A_0 := a_0 + \lambda(b_1 - b_6)\,. \qquad (5.8)$$

- For energy at horizon, the variational formula defines entropy as follows:

$$\delta\Gamma_H = \oint_{S_H} \omega^{01}{}_t \delta H_{01} = 8\kappa A_0\, \delta(\pi r_+^2)\,, \qquad (5.9a)$$

$$\Rightarrow \qquad \delta\Gamma_H = T\delta S\,, \qquad S = 16\pi A_0(\pi r_+^2)\,. \qquad (5.9b)$$

- From the relation $2\delta m = \kappa\delta r_+^2$, we have $\delta E = \delta\Gamma_H$, which confirms the validity of the first law

$$\delta E = T\delta S\,. \qquad (5.10)$$

- The presence of the multiplicative factor $A_0 \neq a_0$ shows that entropy of the Schwarzschild-AdS black hole in PG, as well as the first law, agrees with the corresponding result for diffeomorphism invariant Riemannian theories.
- The GR limit is recovered for $b_1 = b_6 = 0$, $A_0 = a_0$ and $16\pi a_0 = 1$.

- One of the first spherically symmetric solutions of PG has been constructed by Baekler. The metric of the Baekler solution is of the Schwarzschild-AdS form. The ansatz for torsion is assumed to be $O(3)$ invariant:

$$T^0 = T^1 = fb^0 b^1\,, \qquad T^c = -f(b^0 - b^1)b^c\,, \qquad (5.1)$$
$$f := -\frac{m}{r^2 N}\,.$$

- One can now calculate the Riemann-Cartan connection

$$\omega^{01} = -(N' + f)b^0 + fb^1\,, \qquad \omega^{0c} = -fb^c\,,$$
$$\omega^{1c} = \left(\frac{N}{r} - f\right) b^c\,, \qquad \omega^{23} = \frac{\cos\theta}{r\sin\theta} b^3\,. \qquad (5.2)$$

- The curvature 2-form reads

$${}^{(6)}R^{ij} = \lambda b^i b^j\,, \qquad {}^{(4)}R^{Ac} = \frac{\lambda m}{rN^2}(b^0 - b^1)b^c\,.$$

- Dynamics is determined by von der Heyde Lagrangian

$$L_G = a_1 T^{i\star}({}^{(1)}T_i - 2\,{}^{(2)}T_i + {}^{(3)}T_i) + \frac{1}{2}b_1 R^{ij\star}R_{ij}\,, \qquad (5.3)$$

- The field equations imply $2\lambda b_1 = -a_1$, while:

$$\begin{aligned}
H_{01} &= -a_1 b^2 b^3\,, \quad H_{23} = -a_1 b^0 b^1\,,\\
H_{02} &= a_1 b^1 b^3 - a_1\frac{m}{rN^2}(b^0 - b^1)b^3\,,\\
H_{03} &= -a_1 b^1 b^2 + a_1\frac{m}{rN^2}(b^0 - b^1)b^2\,,\\
H_{12} &= -a_1 b^0 b^3 + a_1\frac{m}{rN^2}(b^0 - b^1)b^3\,,\\
H_{13} &= a_1 b^0 b^2 - a_1\frac{m}{rN^2}(b^0 - b^1)b^2\,,\\
H_0 &= -H_1 = 4a_1\frac{m}{r^2N}b^2 b^3\,. \qquad (5.4)
\end{aligned}$$

- Energy of the solution is proportional to m:

$$E = 16\pi a_1 m\,. \tag{5.5}$$

- Entropy is calculated from the variational equation:

$$b^i{}_t \delta H_i = -4\big[N\delta(fr^2)\big]_{r_+} \cdot 4\pi a_1\,,$$
$$\frac{1}{2}\omega^{ij}{}_t \delta H_{ij} = \omega^{01}{}_t \delta H_{01} = (\kappa + \underline{Nf}_\times)\delta r_+^2 \cdot 4\pi a_1$$
$$\frac{1}{2}\delta\omega^{ij} H_{ijt} = \Big[-2fr^2\delta N + 2N\delta(fr^2) - \underline{Nf}_\times \delta r^2\Big]_{r_+} \cdot 4\pi a_1\,.$$

- Summing up these terms we get:

$$\delta\Gamma_H = 8\pi a_1 \kappa \delta r_+^2 = T\delta S\,, \qquad S := 16\pi a_1 \delta(\pi r_+^2)\,. \tag{5.6}$$

- Here, the torsion sector gives a nontrivial contribution to entropy, so *dynamical content* of the result is quite different than in GR.

- Teleparallel gravity (TG) is a subcase of PG, defined by the vanishing Riemann-Cartan curvature, $R^{ij} = 0$. Choosing the related spin connection to vanish, $\omega^{ij} = 0$, the tetrad field remains the only dynamical variable, and torsion takes the form $T^i = db^i$. The general (parity invariant) TG Lagrangian has the form

$$L_T := a_0 T^i \,{}^\star\left(a_1 {}^{(1)}T_i + a_2 {}^{(2)}T_i + a_3 {}^{(3)}T_i\right) . \qquad (5.7a)$$

- In physical considerations, a special role is played by a special *one-parameter family* of TG Lagrangians, defined by a single parameter γ as

$$a_1 = 1\,, \qquad a_2 = -2\,, \qquad a_3 = -1/2 + \gamma\,. \qquad (5.7b)$$

- This family represents a viable gravitational theory for macroscopic matter, empirically indistinguishable from GR.

- Every spherically symmetric solution of GR is also a solution of the one-parameter TG. In particular, this is true for the Schwarzschild-AdS spacetime. Since ${}^{(3)}T_i = 0$, the covariant momentum H^i does not depend on γ:

$$
\begin{aligned}
H^0 &= \frac{2a_0}{r\sin(\theta)}\Big[\cos(\theta)b^1b^3 - 2N\sin(\theta)b^2b^3\Big]\,,\\
H^1 &= \frac{2a_0\cos(\theta)}{r\sin(\theta)}b^0b^3\,, \qquad H^2 = -\frac{2a_0}{r}(rN' + N)b^0b^3\,,\\
H^3 &= \frac{2a_0}{r}(rN' + N)b^0b^2\,. \qquad (5.8)
\end{aligned}
$$

- The energy of the Schwarzschild-AdS solution in TG is

$$E = m\,. \qquad (5.9)$$

- Our approach to entropy yields (integration implicit)

$$
\begin{aligned}
b^i{}_t\delta H_i &= \big[N\delta H_0\big]_{r_+} = -16\pi a_0\big[N\delta(Nr)\big]_{r_+} = 0\,,\\
b^i\delta H_{it} &= \big[b^2\delta H_{2t} + b^3\delta H_{3t}\big]_{r_+} = 8\pi a_0\cdot\kappa\delta(r_+^2)\,,
\end{aligned}
$$

where we used $NN' = \kappa$ and $[N\delta N]_{r_+} = 0$. Thus, with $16\pi a_0 = 1$, one obtains

$$
\delta\Gamma_H = T\delta S\,,\qquad S = \pi r_+^2\,. \tag{5.10a}
$$

The identity $2\delta m = \kappa\delta r_+^2$ confirms the validity of the first law

$$
\delta E = T\delta S\,. \tag{5.11}
$$

- We investigated the notion of entropy in the general (parity preserving) four-dimensional PG. Our approach was based on the idea that black hole entropy can be interpreted as the conserved charge on horizon.
- We constructed the canonical generator G of gauge symmetries as an integral over the spatial section Σ of spacetime, which has to be a regular (differentiable) functional on the phase space. The regularity can be ensured by adding to G a suitable surface term Γ_∞ defined on the boundary of Σ at infinity.
- The form of Γ_∞ is determined by the variational equation and its value defines the asymptotic charge.
- For a black hole solution, Σ has two boundaries, one at infinity and the other at horizon. The condition of regularity of G includes two boundary terms, Γ_∞ and Γ_H.

- The new boundary term Γ_H, defines entropy as the canonical charge on horizon. The regularity of G represents just the first law of black hole thermodynamics.
- We tested our results on three vacuum solutions of the Schwarzschild-AdS type. For Riemannian SAdS geometry as a solution of PG, we found that both energy and entropy differ from the GR expressions by a multiplicative factor. The study of Baekler's solution reveals new dynamical features of PG, the existence of nontrivial contributions to energy and entropy stemming from both the torsion and the curvature sectors. We successfully applied our approach to the teleparallel gravity, where curvature vanishes and entropy is produced solely by torsion.
- An additional test of our approach to black hole entropy can be obtained from the analysis of the Kerr black hole.