Spin connections, local Lorentz transformations and cosmological perturbations in modified teleparallel theories of gravity

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There are different approaches to teleparallel gravity. I prefer to avoid the gauge theory viewpoint.

We work in the tetrad formulation, the metric at each point is associated with the set of tangent vectors via $g_{\mu\nu} = e^A_\mu e^B_\nu \eta_{AB}$ which defines the tetrad fields e^A_μ up to local Lorentz rotations.

Now we can consider every tensor with Latin indices instead of spacetime ones with the relation between the two

$$\mathcal{T}^{A_1,\ldots,A_n}{}_{B_1,\ldots,B_m} \equiv e^{A_1}_{\alpha_1} \cdots e^{A_n}_{\alpha_n} \mathcal{T}^{\alpha_1,\ldots,\alpha_n}{}_{\beta_1,\ldots,\beta_m} e^{\beta_1}_{B_1} \cdots e^{\beta_m}_{B_m}.$$

One can naturally have torsionful connections without non-metricity with the requirement

$$\partial_\mu e^A_\nu + \omega^A{}_{\mu B} e^B_\nu - \Gamma^\alpha_{\mu\nu} e^A_\alpha = 0$$

of vanishing of the "full covariant derivative" of the tetrad.

Even if naive local Lorentz invariance is broken, the construction is to be assumed locally Lorentz invariant under

$$e^A_\mu \longrightarrow \Lambda^A_C e^C_\mu \,, \quad \omega^A{}_{\mu B} \longrightarrow \Lambda^A_C \omega^C{}_{\mu D} (\Lambda^{-1})^D_B - (\Lambda^{-1})^A_C \partial_\mu \Lambda^C_B \,.$$

Nota bene! One can also use formalism of Lagrange multipliers to set curvature (and non-metricity) to zero. This is an important alternative formalism, also to be used for symmetric teleparallelism (non-metricity with neither curvature not trosion)

The condition of vanishing of the "full covariant derivative" of the tetrad is solved straightforwardly to obtain

$$\Gamma^{\alpha}_{\mu\nu} = e^{\alpha}_{A}\left(\partial_{\mu} e^{A}_{\nu} + \omega^{A}{}_{\mu B} e^{B}_{\nu}\right) \equiv e^{\alpha}_{A}\ \mathfrak{D}_{\mu} e^{A}_{\nu}$$

with $\mathfrak{D}_{\mu}$ being the Lorentz-covariant (with respect to the Latin index only) derivative

or another way around

$$\omega^{A}{}_{\mu B} = e^{A}_{\alpha}\Gamma^{\alpha}_{\mu\nu} e^{\nu}_{B} - e^{\nu}_{B}\partial_{\mu} e^{A}_{\nu}$$

In particular, one can find the spin connection $\overset{(0)}{\omega}$ which corresponds to the Levi-Civita connection $\overset{(0)}{\Gamma}(g)$ of a given metric g.

Basically, both $\Gamma^{\alpha}_{\mu\beta}$ and $\omega^{A}{}_{\mu B}$ represent one and the same connection in different disguises. This conclusion is further substantiated by comparing the curvatures for both connections,

$$R^{A}{}_{B\mu\nu}(\omega) = \partial_{\mu}\omega^{A}{}_{\nu B} - \partial_{\nu}\omega^{A}{}_{\mu B} + \omega^{A}{}_{\mu C}\omega^{C}{}_{\nu B} - \omega^{A}{}_{\nu C}\omega^{C}{}_{\mu B}$$

and

$$R^{\alpha}{}_{\beta\mu\nu}(\Gamma) = \partial_{\mu}\Gamma^{\alpha}_{\nu\beta} - \partial_{\nu}\Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\rho}\Gamma^{\rho}_{\nu\beta} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\rho}_{\mu\beta}\,,$$

which after a simple calculation gives

$$R^{\alpha}{}_{\beta\mu\nu}(\Gamma) = e^{\alpha}_{A} R^{A}{}_{B\mu\nu}(\omega) e^{B}_{\beta}\,.$$

In other words, the two Riemann tensors are related by mere change of the types of indices. Therefore, those are one and the same tensor under our conventions which are common for all the tensors we use.

Assuming that $\bigtriangledown_\alpha g_{\mu\nu} = 0$, one can follow the standard textbook derivation of the Levi-Civita connection and prove that

$$\Gamma^\alpha_{\mu\nu} = \overset{(0)}{\Gamma}{}^\alpha_{\mu\nu}(g) + K^\alpha{}_{\mu\nu}$$

where $\overset{(0)}{\Gamma}{}^\alpha_{\mu\nu}(g)$ is the Levi-Civita connection of the metric g, while the tensor K

$$K_{\alpha\mu\nu} = \frac{1}{2}\left(T_{\alpha\mu\nu} + T_{\nu\alpha\mu} + T_{\mu\alpha\nu}\right) = \frac{1}{2}\left(T_{\mu\alpha\nu} + T_{\nu\alpha\mu} - T_{\alpha\nu\mu}\right),$$

is known under the name of contortion.
It is obviously antisymmetric with respect to two indices:

$$K_{\alpha\mu\nu} = -K_{\nu\mu\alpha}.$$

Substituting our connection into the the definition of curvature, we get

$$R^{\alpha}{}_{\beta\mu\nu}(\Gamma) = R^{\alpha}{}_{\beta\mu\nu}(\overset{(0)}{\Gamma}) + \overset{(0)}{\bigtriangledown}_{\mu} K^{\alpha}{}_{\nu\beta} - \overset{(0)}{\bigtriangledown}_{\nu} K^{\alpha}{}_{\mu\beta} + K^{\alpha}{}_{\mu\rho} K^{\rho}{}_{\nu\beta} - K^{\alpha}{}_{\nu\rho} K^{\rho}{}_{\mu\beta}$$

for the Riemann tensor with $\overset{(0)}{\bigtriangledown}_{\mu}$ being the covariant derivative associated to $\overset{(0)}{\Gamma}{}^{\alpha}_{\mu\nu}(g)$.

Making the necessary contractions we obtain the scalar curvature

$$R(\Gamma) = R(\overset{(0)}{\Gamma}) + 2 \overset{(0)}{\bigtriangledown}_{\mu} T^{\mu} + \mathbb{T}$$

where the torsion vector is

$$T_{\mu} \equiv T^{\alpha}{}_{\mu\alpha} = -T^{\alpha}{}_{\alpha\mu},$$

and the torsion scalar can be written in several equivalent ways:

$$
\begin{aligned}
\mathbb{T} &= \frac{1}{2}K_{\alpha\beta\mu}T^{\beta\alpha\mu} - T_\mu T^\mu \\
&= \frac{1}{2}T_{\alpha\beta\mu}S^{\alpha\beta\mu} \\
&= \frac{1}{4}T_{\alpha\beta\mu}T^{\alpha\beta\mu} + \frac{1}{2}T_{\alpha\beta\mu}T^{\beta\alpha\mu} - T_\mu T^\mu
\end{aligned}
$$

with the superpotential

$$S^{\alpha\mu\nu} \equiv K^{\mu\alpha\nu} + g^{\alpha\mu}T^\nu - g^{\alpha\nu}T^\mu$$

which satisfies the antisymmetry condition

$$S^{\alpha\mu\nu} = -S^{\alpha\nu\mu}.$$

In the classical formulation of teleparallel gravity, one uses the Weitzenböck connection given by

$$\overset{\mathfrak{W}}{\omega}{}^{A}{}_{\mu B} = 0$$

or

$$\overset{\mathfrak{W}}{\Gamma}{}^{\alpha}_{\mu\nu} = e^{\alpha}_{a}\partial_{\mu}e^{a}_{\nu}$$

which is obviously curvature-free, $R^{\alpha}{}_{\beta\mu\nu}(\overset{\mathfrak{W}}{\Gamma}) = 0$.

Nota bene!

I. This choice is obviously locally Lorentz breaking. Invariant formulation would be to demand that ω is flat, see below.

II. Another viewpoint on teleparallel gravity is through gauging translations. Then richer structures and bolder claims are possible, such as separation of inertia from gravitation, preferred connection for a given tetrad etc. (see e.g. the book by Aldrovandi and Pereira).
We will stick to our simple definition (roughly, GR in terms of torsion). However, the choice of (flat) spin connection can be important for finiteness of action (variational principle, quantum gravity, etc.).

We can denote the determinant of e_μ^A by $\|e\|$, and see from

$$R(\Gamma) = R(\overset{(0)}{\Gamma}) + 2 \overset{(0)}{\bigtriangledown}_\mu T^\mu + \mathbb{T}$$

that the action

$$S_{\mathfrak{W}} = -\int d^4x \|e\| \cdot \mathbb{T}$$

is equivalent to the action of GR,

$$\int d^4x \sqrt{-g} \cdot R(\overset{(0)}{\Gamma}),$$

modulo the surface term, if the Weitzenböck, or any other inertial, connection is assumed.

We are also interested in equations of motion.
We have the following first order variations for the inverse tetrad, measure, metric and torsion:

$$\begin{aligned}
\delta e^{\mu}_{A} &= -e^{\mu}_{B} e^{\nu}_{A} \delta e^{B}_{\nu}, \\
\delta \|e\| &= \|e\| \cdot e^{\mu}_{A} \delta e^{A}_{\mu}, \\
\delta g_{\mu\nu} &= \eta_{AB} \left(e^{A}_{\mu} \delta e^{B}_{\nu} + e^{A}_{\nu} \delta e^{B}_{\mu} \right), \\
\delta g^{\mu\nu} &= -\left(g^{\mu\alpha} e^{\nu}_{A} + g^{\nu\alpha} e^{\mu}_{A} \right) \delta e^{A}_{\alpha}, \\
\delta_e T^{\alpha}{}_{\mu\nu} &= -e^{\alpha}_{A} T^{\beta}{}_{\mu\nu} \delta e^{A}_{\beta} + e^{\alpha}_{A} \left(\mathfrak{D}_{\mu} \delta e^{A}_{\nu} - \mathfrak{D}_{\nu} \delta e^{A}_{\mu} \right).
\end{aligned}$$

In particular, for the teleparallel equivalent of GR we have $\delta_e S$

$$= - \int d^4x \|e\| \cdot \left(-2S^{\alpha\mu\nu} T_{\alpha\beta\nu} e^{\beta}_{A} \delta e^{A}_{\mu} + \mathbb{T} e^{\mu}_{A} \delta e^{A}_{\mu} - 2S_{\beta}{}^{\mu\alpha} e^{\beta}_{A} \mathfrak{D}_{\alpha} \delta e^{A}_{\mu} \right)$$

with the Lorentz-covariant derivative $\mathfrak{D}$ being equal to the ordinary one, since $\omega^{B}{}_{\alpha A} = 0$ in the Weitzenböck case.

We need to perform integration by parts in the last term which gives

$$2\delta e^A_\mu \cdot \left(\partial_\alpha\left(\|e\|\cdot S_\beta{}^{\mu\alpha} e^\beta_A\right) - \|e\|\cdot \omega^B{}_{\alpha A} S_\beta{}^{\mu\alpha} e^\beta_B\right)$$
$$= 2\|e\|\cdot\left(\overset{(0)}{\nabla}{}_\alpha S_\beta{}^{\mu\alpha} - K^\nu{}_{\alpha\beta} S_\nu{}^{\mu\alpha}\right)\cdot e^\beta_A \delta e^A_\mu$$

where we have used the antisymmetry of S and corrected for the difference between Γ and $\overset{(0)}{\Gamma}$ by the second term on the right hand side. Indeed, due to the antisymmetry of S we have

$$\overset{(0)}{\nabla}{}_\nu S_A{}^{\mu\nu} = \frac{1}{\|e\|}\partial_\nu\left(\|e\| S^{\mu\nu}_A\right) - \overset{(0)}{\omega}{}^B{}_{\nu A} S_B{}^{\mu\nu}$$

and correct for the different connection by noting that $\omega^B{}_{\nu A} - \overset{(0)}{\omega}{}^B{}_{\nu A} = K^B{}_{\nu A}$.

Finally, using the non-degeneracy of tetrads, we get the equations of motion in the form

$$\overset{(0)}{\bigtriangledown}_{\alpha} S_{\beta}{}^{\mu\alpha} - S^{\alpha\mu\nu}\left(T_{\alpha\beta\nu} + K_{\alpha\nu\beta}\right) + \frac{1}{2}\mathbb{T}\delta^{\mu}_{\beta} = 0$$

which can be shown to be equivalent to general relativity by direct substitution of

$$R^{\alpha}{}_{\beta\mu\nu}(\overset{(0)}{\Gamma}) = -\left(\overset{(0)}{\bigtriangledown}_{\mu} K^{\alpha}{}_{\nu\beta} - \overset{(0)}{\bigtriangledown}_{\nu} K^{\alpha}{}_{\mu\beta} + K^{\alpha}{}_{\mu\rho} K^{\rho}{}_{\nu\beta} - K^{\alpha}{}_{\nu\rho} K^{\rho}{}_{\mu\beta}\right)$$

into the Einstein equation,

$$G^{\mu}_{\beta} = 0.$$

What if we covariantise the model by substituting an explicit spin connection?

Variations with respect to the spin connection coefficients can be derived exactly since

$$\delta_\omega T^\alpha{}_{\mu\nu} = \delta\omega^\alpha{}_{\mu\nu} - \delta\omega^\alpha{}_{\nu\mu},$$

is an exact relation for $\delta\omega^\alpha{}_{\mu\nu} \equiv e^\alpha_A e^B_\nu \delta\omega^A{}_{\mu B}$.

Suppose, we want to covariantise the teleparallel action by allowing for an arbitrary spin connection in the torsion scalar,

$$S = -\int d^4x \|e\| \cdot \mathbb{T}(e, \omega),$$

and varying independently with respect to both variables e and ω. We have

$$\delta_\omega S = -\int d^4x \|e\| \cdot \left(T^{\mu}{}_{\alpha\nu} + 2T_\nu \delta^\mu_\alpha \right) \delta\omega^{\alpha}{}_{\mu}{}^{\nu}.$$

The equation of motion is

$$T^{\mu}{}_{\alpha\nu} + T_\nu \delta^\mu_\alpha - T_\alpha \delta^\mu_\nu = 0$$

which (in dimension $d \neq 2$) entails $T_\mu = 0$ upon tracing, and totally

$$T^{\mu}{}_{\alpha\nu} = 0.$$

It does not give the desired result!

A better idea would be to vary the spin connection in the inertial class only. The latter can be imposed by demanding

$$\omega^{A}{}_{\mu B} = -(\Lambda^{-1})^{A}_{C}\partial_{\mu}\Lambda^{C}_{B}$$

where Λ is an arbitrary Lorentz matrix and varying

$$S_{\mathfrak{W}'} = -\int d^4x \|e\| \cdot \mathbb{T}(e, \omega(\Lambda))$$

with respect to e and Λ.

Literally it means that there exists a frame in which $\omega = 0$ (Weitzenböck), however one is allowed to make a local Lorentz rotation by an arbitrary matrix field $\Lambda^{A}_{B}(x)$ whose values belong to Lorentz group.

Explicit calculations are given in
Alexey Golovnev, Tomi Koivisto, Marit Sandstad.
On the covariance of teleparallel gravity theories.
Classical and Quantum Gravity 34 (2017) 145013
https://arxiv.org/abs/1701.06271

However, the essence is very simple. Varying the spin connection with fixed tetrads does not change the Levi-Civita connection, while we know that in any case

$$\delta_{\Lambda}\mathbb{T} = \delta_{\Lambda} R(\omega) - 2 \overset{(0)}{\bigtriangledown}_{\mu}(\delta_{\Lambda} T^{\mu})$$

where $\delta_{\Lambda}(...) = \delta_{\omega}(...) \cdot \delta_{\Lambda}\omega$.
Since $R(\omega(\Lambda)) \equiv 0$, the variation $\delta_{\omega} S_{\mathfrak{W}\mathfrak{Y}'}$ is a surface term and does not produce any new equation of motion. The model, though locally Lorentz covariant, is then equivalent to teleparallel gravity.

It would be interesting to try making modifications of GR in the teleparallel framework. One very popular example is $f(T)$ gravity.

Note that there is not even a universal agreement in the community about the number of degrees of freedom in this model.

The covariantisation procedure works differently in generalised teleparallel gravities. Since the dependence on the spin connection in generalised models cannot be reduced to a surface term, the variation δ_ω produces non-trivial equations of motion.

However, admissible variations of ω in the inertial class amount to local Lorentz transformations. Note also that a covariantised action is, by definition, identically invariant under simultaneous local Lorentz transformation of the spin connection and the tetrad (and other non-trivially transforming fields if there are some). Therefore, the stationarity of the action under local Lorentz transformations of the spin connection is equivalent to that under local Lorentz rotations of tetrads. The latter is already ensured given that the equations of motion for the tetrad are satisfied since the local Lorentz rotation is nothing but a special class of variations of the tetrad.

$$e^A_\mu \longrightarrow \Lambda^A_C e^C_\mu \,, \quad \omega^A{}_{\mu B} \longrightarrow \Lambda^A_C \omega^C{}_{\mu D} (\Lambda^{-1})^D_B - (\Lambda^{-1})^A_C \partial_\mu \Lambda^C_B \,.$$

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Let us consider an $f(T)$ model with inertial spin connection,

$$S_{f(T)} = -\int d^4x \|e\| \cdot f\left(\mathbb{T}(e, \omega(\Lambda))\right).$$

We want to derive equations of motion.

Making the variation with respect to the tetrad gives

$$f_T \overset{(0)}{G}_{\mu\nu} + f_{TT} S_{\mu\nu\alpha} \partial^\alpha \mathbb{T} + \frac{1}{2} (f - f_T \mathbb{T})\, g_{\mu\nu} = 0.$$

Unlike in the TEGR case, this equation has a non-trivial antisymmetric part

$$T^{\alpha\mu\nu} \partial_\alpha f_T(\mathbb{T}) + T^\nu \partial^\mu f_T(\mathbb{T}) - T^\mu \partial^\nu f_T(\mathbb{T}) = 0$$

which reflects the non-invariance under local Lorentz rotations of tetrads. Variation with respect to the purely inertial spin connection gives the same result.

Nota bene! The equation of motion in our paper was incorrect. Correct equation dates back to an old paper by Li, Sotiriou and Barrow.

Let us also make a general comment on the structure of equations. The broken local Lorentz invariance implies that the Bianchi identities do not hold automatically. Indeed, if we define $\mathfrak{T}^{\mu\nu}$ via

$$\frac{\delta S}{\delta e^A_\mu} \equiv \|e\| \mathfrak{T}^{\mu\nu} e^B_\nu \eta_{AB},$$

then invariance of the action under $e^a_\mu \longrightarrow e^A_\mu - e^A_\nu \partial_\mu \zeta^\nu - \zeta^\nu \partial_\nu e^A_\mu$ leads to

$$\frac{1}{\|e\|} \partial_\mu \left(\|e\| \mathfrak{T}^\mu_\nu \right) - \mathfrak{T}^\beta_\alpha e^\alpha_A \partial_\nu e^A_\beta = 0$$

which can easily be transformed (using $K_{\alpha\mu\beta} - T_{\alpha\mu\beta} = -K_{\mu\beta\alpha}$) into

$$\overset{(0)}{\bigtriangledown}_\mu \mathfrak{T}^{\mu\nu} + K^{\alpha\nu\beta} \mathfrak{T}_{\alpha\beta} = 0.$$

When the local Lorentz invariance is satisfied, invariance under $e_\mu^A \longrightarrow \Lambda_B^A e_\mu^B$ implies that $\mathfrak{T}^{\mu\nu}$ is symmetric, and by virtue of antisymmetry of contortion tensor, the usual Bianchi identities are restored. In $f(T)$ this is not the case. However, the antisymmetric part of equations requires that the antisymmetric part of $\mathfrak{T}^{\mu\nu}$ vanishes, and after that the Bianchi identities are in operation again. We will see below that this condition allows one to determine ζ. However, cases with fermions and/or spin-density should be considered with care.

$f(T)$ is quite popular for cosmology.
What do we know about cosmological perturbations?
There used to be a lot of confusion.

We develop an $f(T)$-type FRW cosmology

$$ds^2 = a^2(\tau)\left(-d\tau^2 + dx^i dx^i\right)$$

in terms of the following tetrad ansatz:

$$e_\mu^A = a(\tau) \cdot \delta_\mu^A.$$

Nota bene! A tetrad choice for the given metric has been done which is not innocuous in the context of $f(T)$, but in this case seems reasonable.

Let's have $\mu = 0, i$ and $A = \emptyset, a$ for time and space components.

Let us consider the following standard parametrisation of the metric fluctuations:

$$\begin{aligned} g_{00} &= -a^2(\tau)\cdot(1+2\phi) \\ g_{0i} &= a^2(\tau)\cdot(\partial_i\zeta + v_i) \\ g_{ij} &= a^2(\tau)\cdot\left((1-2\psi)\delta_{ij} + 2\partial^2_{ij}\sigma + \partial_i c_j + \partial_j c_i + h_{ij}\right) \end{aligned}$$

with four scalars ϕ, ψ, ζ, σ, two divergenceless vectors v_i, c_i, and divergenceless traceless tensor h_{ij}.

One can choose the following parametrisation for the tetrad:

$$
\begin{aligned}
e_0^{\emptyset} &= a(\tau)\cdot(1+\phi)\\
e_i^{\emptyset} &= 0\\
e_0^{a} &= a(\tau)\cdot(\partial_a\zeta + v_a)\\
e_j^{a} &= a(\tau)\cdot\left((1-\psi)\delta_j^a + \partial_{aj}^2\sigma + \partial_j c_a + \frac{1}{2}h_{aj}\right).
\end{aligned}
$$

However, in $f(T)$ we must also consider local Lorentz rotations of this choice (or variations of the spin connection).

Finally, we can use an ansatz

$$
\begin{aligned}
e_0^{\emptyset} &= a(\tau)\cdot(1+\phi) \\
e_i^{\emptyset} &= a(\tau)\cdot(\partial_i\beta+u_i) \\
e_0^{a} &= a(\tau)\cdot(\partial_a\zeta+v_a) \\
e_j^{a} &= a(\tau)\cdot\left((1-\psi)\delta_j^a+\partial_{aj}^2\sigma+\epsilon_{ajk}\partial_k s+\partial_j c_a+\epsilon_{ajk}w_k+\frac{1}{2}h_{aj}\right).
\end{aligned}
$$

with the metric components given by

$$
\begin{aligned}
g_{00} &= -a^2(\tau)\cdot(1+2\phi) \\
g_{0i} &= a^2(\tau)\cdot\left(\partial_i\left(\zeta-\beta\right)+v_i-u_i\right) \\
g_{ij} &= a^2(\tau)\cdot\left((1-2\psi)\delta_{ij}+2\partial_{ij}^2\sigma++\partial_i c_j+\partial_j c_i+h_{ij}\right).
\end{aligned}
$$

and new components given by scalar β, pseudoscalar s, vector u_i, and pseudovector w_j.

Under infinitesimal diffeomorphisms $x^\mu \to x^\mu + \xi^\mu(x)$ with ξ^0 and $\xi^i \equiv \partial_i \xi + \tilde{\xi}_i$, one can simply derive the following transformation laws:

$$\begin{aligned}
\phi &\longrightarrow \phi - \xi^{0\prime} - H\xi^0 \\
\psi &\longrightarrow \psi + H\xi^0 \\
\sigma &\longrightarrow \sigma - \xi \\
\beta &\longrightarrow \beta - \xi^0 \\
\zeta &\longrightarrow \zeta - \xi' \\
c_i &\longrightarrow c_i - \tilde{\xi}_i \\
v_i &\longrightarrow v_i - \tilde{\xi}_i'.
\end{aligned}$$

Guage invariant combinations are obvious.

Our **gauge choice**: $\sigma = 0$ and $\beta = \zeta$ (conformal Newtonian gauge), and $c_i = 0$.

Now we are in a position to vary the equation (with matter contribution)

$$f_T \overset{(0)}{G}_{\mu\nu} + f_{TT} S_{\mu\nu\alpha} \partial^\alpha \mathbb{T} + \frac{1}{2}\left(f - f_T \mathbb{T}\right) g_{\mu\nu} = 8\pi G \cdot \Theta_{\mu\nu}$$

since all the quantities are known, either from the standard cosmological perturbation theory or from the torsion components above.

The antisymmetric part of equations is relatively simple. Suppose that $f_{TT} \neq 0$ and that the energy-momentum tensor of matter is symmetric. In this case we have

$$(S_{\mu\nu\alpha} - S_{\nu\mu\alpha})\,\partial^{\alpha}\mathbb{T} = 0.$$

One can easily see that it boils down to

$$(T_{\alpha\mu\nu} + g_{\alpha\mu}T_{\nu} - g_{\alpha\nu}T_{\mu})\,\partial^{\alpha}\mathbb{T} = 0,$$

our antisymmetric part of equation from previous work.

For symmetric part of equations, let us denote

$$Q_{\mu\nu} \equiv \frac{1}{2}\left(S_{\mu\nu\alpha} + S_{\nu\mu\alpha}\right)\partial^{\alpha}\mathbb{T}$$

where we easily see that

$$S_{\mu\nu\alpha} + S_{\nu\mu\alpha} = T_{\mu\nu\alpha} + T_{\nu\mu\alpha} + 2g_{\mu\nu}T_{\alpha} - \left(g_{\alpha\mu}T_{\nu} + g_{\alpha\nu}T_{\mu}\right).$$

At the background level the only non-trivial components are spatial

$$Q^i_j = -\frac{24H^2}{a^4}(H' - H^2)\delta^i_j.$$

One can write the symmetric part of equation as

$$f_T \overset{(0)}{G^\mu_\nu} + f_{TT}Q^\mu_\nu + \frac{1}{2}\left(f - f_T\mathbb{T}\right)\delta^\mu_\nu = 8\pi G\Theta^\mu_\nu.$$

We see that at the background level only $T_{i0j} = -T_{ij0}$ components are non-vanishing. The only non-zero components of the superpotential at the background level are

$$S_{i0j} = -S_{ij0} = -2a^2 H\delta_{ij},$$

and one can easily check that

$$\mathbb{T} \equiv \frac{1}{2} S^{\alpha\mu\nu} T_{\alpha\mu\nu} = \frac{6}{a^2} H^2,$$

and all background equations from the previous works can be reproduced.

The tensor sector is simple.
It is easy to see that, under transverse and traceless condition, the only non-zero torsion components are

$$T_{ijk} = \frac{a^2}{2}\left(\partial_j h_{ik} - \partial_k h_{ij}\right),$$

$$T_{i0j} = a^2\left(H\delta_{ij} + \frac{1}{2}\left(h'_{ij} + 2Hh_{ij}\right)\right),$$

and both $\delta\mathbb{T} = 0$ and $\delta T_\mu = 0$.
The antisymmetric part of equation $T_{0ij} = 0$ is satisfied identically.

In the symmetric part we have

$$Q^i_j = \left(T^i{}_{j0} + \delta^i_j T_0\right) \partial^0 \mathbb{T} = \frac{12H(H' - H^2)}{a^4} \left(-2H\delta_{ij} + \frac{1}{2} h'_{ij}\right)$$

which via $a^2 \delta G^i_j = \frac{1}{2} \left(h''_{ij} + 2Hh'_{ij} - \triangle h_{ij}\right)$ leads to

$$f_T h''_{ij} + 2H \left(f_T + \frac{6f_{TT}(H' - H^2)}{a^2}\right) h'_{ij} - f_T \triangle h_{ij} = 0$$

for an ideal fluid.

Our ansatz is

$$
\begin{aligned}
e_0^{\emptyset} &= a(\tau)\cdot(1+\phi)\\
e_i^{\emptyset} &= a(\tau)\cdot(\partial_i\beta+u_i)\\
e_0^{a} &= a(\tau)\cdot(\partial_a\zeta+v_a)\\
e_j^{a} &= a(\tau)\cdot\left((1-\psi)\delta_j^a+\partial_{aj}^2\sigma+\epsilon_{ajk}\partial_k s+\partial_j c_a+\epsilon_{ajk}w_k+\frac{1}{2}h_{aj}\right).
\end{aligned}
$$

Let's look at vectors in $c=0$ gauge.

In the vector sector, we easily compute the following torsion components:

$$
\begin{aligned}
T_{0ij} &= a^2 \left(\partial_j u_i - \partial_i u_j\right) \\
T_{00i} &= a^2 \partial_i \left(-u_i' + H(v_i - u_i)\right) \\
T_{ijk} &= a^2 \cdot \left(\epsilon_{ikl}\partial_j w_l - \epsilon_{ijl}\partial_k w_l\right) \\
T_{i0j} &= a^2 \left(H\delta_{ij} + \epsilon_{ijk} w_k' - \partial_j v_i\right)
\end{aligned}
$$

and the torsion vector

$$T_0 = 3H,$$

$$T_i = \epsilon_{ijk}\partial_j w_k.$$

and see that at the linear order

$$\delta\mathbb{T} = 0.$$

(The latter was to be expected since we cannot construct a scalar out of vectors.)

The antisymmetric part of equations $(T_{\alpha\mu\nu} + g_{\alpha\mu} T_\nu - g_{\alpha\nu} T_\mu)\,\partial^\alpha \mathbb{T} = 0$ then boils down to

$$T_{0\mu\nu} + g_{0\mu} T_\nu - g_{0\nu} T_\mu = 0.$$

With spatial indices we get

$$\partial_j u_i - \partial_i u_j = 0$$

which, after taking divergence, implies

$$\triangle u_i = 0$$

and, in perturbation theory, should be solved as

$$u = 0.$$

The mixed indices case gives

$$u_i' + 2H(v_i - u_i) + \epsilon_{ijk}\partial_j w_k = 0$$

which constrains w as

$$\epsilon_{ijk}\partial_j w_k = -2Hv_i.$$

Here we have two independent equations for two independent components of w.

Now let us look at the symmetric part:

$$Q_{\nu\mu} = \left(\frac{1}{2}\left(T_{\mu\nu\alpha} + T_{\nu\mu\alpha}\right) + g_{\mu\nu} T_{\alpha} - \frac{1}{2}\left(g_{\alpha\mu} T_{\nu} + g_{\alpha\nu} T_{\mu}\right)\right) \partial^{\alpha}\mathbb{T}.$$

For mixed indices we easily find that

$$Q_{0i} = -\frac{6H(H' - H^2)}{a^2}\left(u_i' + 2H(v_i - u_i) + \epsilon_{ijk}\partial_j w_k\right)$$

which vanishes under the antisymmetric part of equations.

Therefore, this part of Einstein equations is not modified:

$$f_T \bigtriangleup v_i = 16\pi G a(\rho + p)\mathfrak{u}_i$$

where $\mathfrak{u}$ is the vortical part of ideal fluid velocity, and we have used $u = 0$ to write simply v instead of the metric perturbation $v - u$.

Analogously we find

$$\delta Q_{ij} = -\frac{6H(H' - H^2)}{a^2}\left(\partial_i v_j + \partial_j v_i\right)$$

which with $a^2 \delta G^i_j = -\frac{1}{2}\left(\partial_i v_j + \partial_j v_i\right)' - H\left(\partial_i v_j + \partial_j v_i\right)$ gives

$$f_T \cdot v_i' + 2\left(f_T H + \frac{6 f_{TT} H(H' - H^2)}{a^2}\right) v_i = 0$$

in case of perfect fluid matter.

Let us consider the pseudoscalar variaton given by $e_i^a = a(\tau)\left(\delta_i^a + \epsilon_{aij}\partial_j s\right)$. The only non-zero components of the torsion tensor and the torsion vector are

$$T_{i0j} = -T_{ij0} = a^2\left(H\delta_{ij} + \epsilon_{ijk}\partial_k s'\right),$$

$$T_0 = 3H.$$

One can easily see that this variation of the tetrad does not contribute to the linearised equations of motion at all.

True "remnant symmetry"?
Unlike other Lorentzian modes (vector, pseudovector and scalar, see below) which are not dynamical but constrained.

Our ansatz is

$$
\begin{aligned}
e_0^{\emptyset} &= a(\tau)\cdot(1+\phi) \\
e_i^{\emptyset} &= a(\tau)\cdot(\partial_i\beta + u_i) \\
e_0^{a} &= a(\tau)\cdot(\partial_a\zeta + v_a) \\
e_j^{a} &= a(\tau)\cdot\left((1-\psi)\delta_j^a + \partial^2_{aj}\sigma + \epsilon_{ajk}\partial_k s + \partial_j c_a + \epsilon_{ajk} w_k + \frac{1}{2}h_{aj}\right).
\end{aligned}
$$

Let's look at scalars in $\sigma = 0$, $\beta = \zeta$ gauge.
In terms of metric, this is conformal Newtonian gauge.

We compute the first order perturbations of the torsion tensor

$$T_{\alpha\mu\nu} \equiv e^B_\alpha \eta_{AB} \left(\partial_\mu e^A_\nu - \partial_\nu e^A_\mu\right)$$

to the get the following components:

$$\begin{aligned} T_{0ij} &= 0 \\ T_{00i} &= a^2 \partial_i \left(\phi - \zeta'\right) \\ T_{ijk} &= a^2 \cdot \left(\delta_{ij}\partial_k \psi - \delta_{ik}\partial_j \psi\right) \\ T_{i0j} &= a^2 \left[H\delta_{ij} - \partial^2_{ij}\zeta - \delta_{ij}\left(2H\psi + \psi'\right)\right] \end{aligned}$$

up to the linear order. Note that T_{i0j} is symmetric under $i \leftrightarrow j$.

Let us also find the torsion vector

$$T_i = e_A^\mu \left(\partial_i e_\mu^A - \partial_\mu e_i^A\right) = g^{\mu\nu} T_{\mu i \nu} = \partial_i \left(\phi - \zeta' - 2\psi\right)$$

and analogously

$$T_0 = g^{\mu\nu} T_{\mu 0 \nu} = 3H - \triangle\zeta - 3\psi',$$

and the torsion scalar

$$\delta\mathbb{T} = -\frac{4H}{a^2} \left(\triangle\zeta + 3H\phi + 3\psi'\right).$$

Let us start from the antisymmetric part of equations.
One can easily check that $(S_{ij\alpha} - S_{ji\alpha})\, \partial^{\alpha}\mathbb{T}$ vanishes identically in the linear order.
And the mixed components give

$$\partial_i \left(H \triangle \zeta + 3H^2\phi + 3H\psi' - 3H'\psi + 3H^2\psi \right) = 0.$$

which can be solved as

$$\triangle\zeta = -3\left(\psi' + H\phi - \frac{H' - H^2}{H}\psi\right).$$

We see that the antisymmetric part of perturbation equations makes perfect sense making the (essentially Lorentz) variable ζ constrained. Now, we have the usual number of equations for the usual number of variables for the symmetric part.

It is easy to see that at the linear level $\delta Q_0^0 = 0$ and the equation

$$f_T \delta \overset{(0)}{G_0^0} + f_{TT} \left(\overset{(0)}{G_0^0} - \frac{1}{2}\mathbb{T} \right) \delta\mathbb{T} = -8\pi G \delta\rho$$

yields the result

$$f_T \triangle\psi - 3H \left(f_T + \frac{12H^2}{a^2} f_{TT} \right) (\psi' + H\phi) - 12 \frac{f_{TT} H^3}{a^2} \triangle\zeta = 4\pi G a^2 \delta\rho$$

where we have used

$$a^2 \overset{(0)}{G_0^0} = -3H^2 - 2 \triangle \psi + 6H \left(\psi' + H\phi\right) \quad \mathrm{a}nd \quad \delta\Theta_0^0 = -\delta\rho.$$

For the mixed components we have

$$\delta Q_i^0 = -\frac{4H^2}{a^4}\partial_i\left(\triangle\zeta + 3H\phi + 3\psi' + 3\frac{H' - H^2}{H}\psi\right)$$

which, with $a^2\delta \overset{(0)}{G_i^0} = -2\partial_i\left(\psi' + H\phi\right)$ and $\delta\Theta_i^0 = -\frac{1}{a}(\rho + p)\partial_i\mathfrak{u}$,

brings the equation $f_T\delta \overset{(0)}{G_i^0} + f_{TT}\delta Q_i^0 = 8\pi G\delta\Theta_i^0$ into the form

$$f_T\left(\psi' + H\phi\right) + \frac{2H^2}{a^2}f_{TT}\left(\triangle\zeta + 3H\phi + 3\psi' + 3\frac{H' - H^2}{H}\psi\right) = 4\pi Ga(\rho + p)\delta\mathfrak{u}$$

where $\mathfrak{u}$ is the velocity potential, and using our solution for ζ it can be brought to a nicer form of

$$f_T\left(\psi' + H\phi\right) + \frac{12H\left(H' - H^2\right)f_{TT}}{a^2}\psi = 4\pi Ga(\rho + p)\delta\mathfrak{u}.$$

As usual, it constrains the velocity potential.

The spatial components are complicated. Assuming no anisotropic stress, their ∂^2_{ij} part gives

$$f_T(\phi - \psi) + 12 f_{TT} H(H' - H^2)\zeta = 0.$$

It is interesting to note that we have gravitational slip

$$\phi - \psi = -\frac{12 f_{TT} H(H' - H^2)}{f_T}\zeta$$

even without anisotropic stress.
Moreover, given our solution for ζ

$$\triangle\zeta = -3\left(\psi' + H\phi - \frac{H' - H^2}{H}\psi\right),$$

it might be very big a slip for superhorizon modes, unless very close to de Sitter.

For the remaining piece of information, one can take trace of the spatial equation to get

$$
\begin{aligned}
f_T \left(\psi'' + H(2\psi + \phi)' + (H^2 + 2H')\phi + \frac{1}{3} \triangle (\phi - \psi)\right) \\
+ \frac{4f_{TT}}{a^2} \left(H^2 \triangle \zeta' + H(3H' - H^2) \triangle \zeta\right) \\
+ \frac{12f_{TT}}{a^2} \left(H^2\psi'' + H(3H' - H^2)\psi' + H^3\phi' + H^2(5H' - 2H^2)\phi\right) \\
+ \frac{48f_{TTT}H^3(H' - H^2)}{a^4} \left(\triangle\zeta + 3H\phi + 3\psi'\right) = 4\pi Ga^2\delta p
\end{aligned}
$$

Precisely as we have done above for the mixed components, one can substitute the solution for $\triangle\zeta$ from the antisymmetric part into the temporal and diagonal spatial components (or even into the intermediate steps of derivations for then the way to them will become much shorter) and get

$$f_T\left(\triangle\psi - 3H(\psi' + H\phi)\right) - \frac{36 f_{TT} H^2 (H' - H^2)}{a^2}\psi = 4\pi G a^2 \delta\rho$$

and

$$f_T\left(\psi'' + H(2\psi+\phi)' + (H^2 + 2H')\phi + \frac{1}{3}\triangle(\phi - \psi)\right) + \frac{12 f_{TT}}{a^2}\Bigg(H(H' - H^2)\psi' + \left(HH'' + 2H'^2 - 5H^2H' + H^4\right)\psi + H^2(H' - H^2)\phi\Bigg) + \frac{144 f_{TTT} H^2 (H' - H^2)^2}{a^4}\psi = 4\pi G a^2 \delta p$$

If, for a given mode with a wavenumber k, we solve for the gravitational slip as (see antisymmetric and spatial non-diagonal components)

$$-k^2 f_T\,(\phi - \psi) = 36 f_{TT}(H' - H^2)\left(H\psi' + H^2\phi - \left(H' - H^2\right)\psi\right),$$

substitute ϕ as a function of ψ and combine two equations for an adiabatic mode by $\delta p = c_s^2\delta\rho$, we will get a second order equation for ψ, much the same way as in GR, though not that nice.
Note that in the limit of $k \to \infty$ the gravitational slip vanishes, and the problem might be tractable.

After careful calculations, there are no new dynamical modes in linear perturbations!
(5+1=6; there are 5 constrained variables and 1 dropping off any equations ("remnant symmetry"?))

Details are given in
A. Golovnev, T. Koivisto https://arxiv.org/abs/1808.05565

A set of personal opinions

- Dynamics of $f(T)$ and other modified teleparallel models are poorly understood yet. The new modes require further investigation.
- Even though introduction of flat spin connection does not change the model bringing a bunch of new variables, it might be interesting and hopefully productive to look from this point of view.
- Linear perturbations around spatially flat FRW are not very difficult. New dynamical modes do not show up. Validity of linear approximation and viability of the model are very questionable.
- Topics around remnant symmetry are not well understood. In general, analysis in terms of Lorentz matrices as new fields is also a very interesting approach.
- In my opinion, $f(T)$ gravity is pathological. However, it would be very interesting to understand the details of pathology.

Thank you for your attention!