From 3D torus with H-flux to torus with R-flux and back

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Outline of the talk

Open string noncommutativity

- For constant background open bosonic string noncommutativity comes from **boundary conditions** if **Kalb-Ramond** field is nonzero.
- It could be shown either by solving equations of motion using Fourier expansion or by canonical methods. In canonical formalism boundary conditions are treated like **constraints**. Because it iturns out that they are of the second class, we can either introduce Dirac brackets or solve the constraints.
- Parameter of noncommutativity is constant, so, there is no nonassociativity.
- Closed string in constant background - there are no boundary conditions, coordinates are commutative.

T-duality and closed strings

- In order to obtain nonassociativity - try with closed strings and coordinate dependent background.
- Weakly curved background - constant metric and linearly coordinate dependent Kalb-Ramond field with infinitesimal field stregth. This choice of background fields satisfies the space-time field equations (consistency conditions).
- Tool - generalized T-duality procedure. Beside two standard steps in Buscher procedure, there is one more - introducing of invariant (covariant) coordinate.
- The simplest case for analysis is 3D torus with flat metric and Kalb-Ramond field with one nonzero component $B_{xy} = Hz$.

T-duality and closed strings

- T-dualization procedure gives the relation between initial and T-dual coordinates - transformation laws.
- We eliminate time derivatives of the initial coordinates using expressions for momenta of the initial theory and get transformation in **canonical form.**
- Because initial theory is geometrical one, their coordinates and momenta satisfy standard algebra

$$\{x^\mu, x^\nu\} = 0\,, \quad \{\pi_\mu, \pi_\nu\} = 0\,, \quad \{x^\mu(\sigma), \pi_\nu(\bar{\sigma})\} = \delta^\mu{}_\nu \delta(\sigma - \bar{\sigma})\,.$$

Buscher procedure of T-dualization

- Global shift symmetry exists $x^a \to x^a + b^a$, where index a is subset of μ.
- We introduce gauge fields $v^a_\pm$ and covariant derivatives $D_\pm x^\mu \equiv \partial_\pm x^a + v^a_\pm$.
- Additional term in the action

$$S_{gauge}(y, v_\pm) = \frac{1}{2}\kappa \int_\Sigma d^2\xi \left(v^a_+ \partial_- y_a - \partial_+ y_a v^a_- \right) ,$$

  where y_a is Lagrange multiplier. It makes $v^a_\pm$ to be unphysical degrees of freedom.
- On the equations of motion for y_a we get initial action, while, fixing x^a to zero, on the equations of motion for $v^a_\pm$ we get T-dual action.

Generalized T-dualization and invariant coordinates

- Buscher procedure works along directions on which background fields do not depend - isometry directions.
- In the case of coordinate dependence - introduction of invariant coordinate in the form

$$x^a_{inv} \equiv \int d\xi^\alpha D_\alpha x^a = x^a(\xi) - x^a(\xi_0) + \Delta V^a \,,$$

  where ξ^α parametrize the world-sheet, while ΔV^a is defined as line integral of gauge field.
- Fixing the gauge $x^a_{inv} \to \Delta V^a$.

Model

- The action is of the form

$$S = \kappa \int_{\Sigma} d^2\xi \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu ,$$

$$G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B_{\mu\nu} = \begin{pmatrix} 0 & Hz & 0 \\ -Hz & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here $\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}$, $\partial_\pm = \partial_\tau \pm \partial_\sigma$ is world-sheet derivative with respect to the light-cone coordinates $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$ and

$$x^\mu = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

T-dualization along x

- Gauge fixed action is of the form

$$\begin{aligned} S_{fix} &= \kappa \int d^2\xi \left[\frac{1}{2}\left(v_+ v_- + \partial_+ y \partial_- y + \partial_+ z \partial_- z\right) + v_+ H z \partial_- y \right. \\ &- \left. \partial_+ y H z v_- + \frac{1}{2} y_1 \left(\partial_+ v_- - \partial_- v_+\right)\right] . \end{aligned}$$

- Combining equations of motion for gauge fields $v_\pm$ and Lagrange multiplier y_1, we get T-dual transformation law

$$\partial_\pm x \cong \pm \partial_\pm y_1 \pm 2 H z \partial_\pm y \,,$$

where $\cong$ denotes T-duality relation.

T-dual theory and canonical form of transformation law

- After T-dualization along x we get twisted torus geometry

$$
{}_xB_{\mu\nu} = 0\,, \quad {}_xG_{\mu\nu} = \begin{pmatrix} 1 & 2Hz & 0 \\ 2Hz & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

- Using expression for initial canonical momentum $\pi_x = \kappa(\dot{x} - 2Hzy')$, **in approximation linear in** H, we get T-dual transformation law in canonical form

$$
\pi_x \cong \kappa y_1'\,.
$$

- T-dual theory is geometrical and commutative one.

T-dualization along y

- T-dual background fields are

$$
{}_{xy}B_{\mu\nu} = \begin{pmatrix} 0 & -Hz & 0 \\ Hz & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -B_{\mu\nu} \,, \quad {}_{xy}G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .
$$

- Canonical form of the T-dual transformation law is

$$
\pi_y \cong \kappa y_2' \,.
$$

- After 2 steps we get commutative, locally well defined nongeometrical theory (Q-flux)[arXiv:1211.6437].

T-dualization along z

- The generalized T-dualization procedure - introduction of the invariant coordinate as line integral

$$z^{inv} = \int_P d\xi^\alpha D_\alpha z = \int_P d\xi^+ D_+ z + \int_P d\xi^- D_- z = z(\xi) - z(\xi_0) + \Delta V$$

where $\Delta V = \int_P d\xi^\alpha v_\alpha = \int_P (d\xi^+ v_+ + d\xi^- v_-)$.

- Variation of the term with ΔV

$$\delta_v \left(-2\kappa \int d^2\xi \varepsilon^{\alpha\beta} H \partial_\alpha y_1 \partial_\beta y_2 \Delta V \right) = \kappa \int d^2\xi \left(\beta^+ \delta v_+ + \beta^- \delta v_- \right)$$

where $\beta^\pm = \pm\frac{1}{2} H (y_1 \partial_\mp y_2 - y_2 \partial_\mp y_1)$.

T-dual tranformation law and background

- The form of the law is

$$y_3' \cong \frac{1}{\kappa}\pi_z - H(xy' - yx')\,.$$

- The background fields

$${}_{xyz}B_{\mu\nu} = \begin{pmatrix} 0 & -H\Delta\tilde{y}_3 & 0 \\ H\Delta\tilde{y}_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, {}_{xyz}G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\partial_\pm y_3 \equiv \pm\partial_\pm \tilde{y}_3$.

Full T-dualized theory

- Full T-dualized theory is **nonlocal** (ΔV causes it) - R flux theory.
- If it holds

$$\{A'(\sigma), B'(\bar{\sigma})\} = U'(\sigma)\delta(\sigma - \bar{\sigma}) + V(\sigma)\delta'(\sigma - \bar{\sigma}) \,,$$

then

$$\{A(\sigma), B(\bar{\sigma})\} = -[U(\sigma) - U(\bar{\sigma}) + V(\bar{\sigma})]\theta(\sigma - \bar{\sigma}) \,.$$

Here

$$\theta(x) = \int_0^x d\eta \delta(\eta) = \frac{1}{2\pi}\left[x + 2\sum_{n\geq 1}\frac{1}{n}\sin(nx)\right] = \begin{cases} 0 & \text{if } x = 0 \\ 1/2 & \text{if } 0 < x < 2\pi \\ 1 & \text{if } x = 2\pi \end{cases}$$

Noncommutativity relations

- From

$$\{y_1'(\sigma), y_3'(\bar\sigma)\} \cong \frac{2}{\kappa} H y'(\sigma)\delta(\sigma-\bar\sigma) + \frac{1}{\kappa} H y(\sigma)\delta'(\sigma-\bar\sigma)\,,$$

$$\{y_2'(\sigma), y_3'(\bar\sigma)\} \cong -\frac{2}{\kappa} H x'(\sigma)\delta(\sigma-\bar\sigma) - \frac{1}{\kappa} H x(\sigma)\delta'(\sigma-\bar\sigma)\,,$$

we obtain

$$\{y_1(\sigma), y_3(\bar\sigma)\} \cong -\frac{H}{\kappa}\left[2y(\sigma) - y(\bar\sigma)\right]\theta(\sigma-\bar\sigma)\,,$$

$$\{y_2(\sigma), y_3(\bar\sigma)\} \cong \frac{H}{\kappa}\left[2x(\sigma) - x(\bar\sigma)\right]\theta(\sigma-\bar\sigma)\,.$$

Noncommutativity relations

- For $\sigma = \bar{\sigma}$, T-dual theory is commutative one.
- Other choices gives noncommutativity. For special choice $\sigma - \bar{\sigma} = 2\pi$ we have

$$\{y_1(\sigma + 2\pi), y_3(\sigma)\} \cong -\frac{H}{\kappa}\left[4\pi N_y + y(\sigma)\right] ,$$

$$\{y_2(\sigma + 2\pi), y_3(\sigma)\} \cong \frac{H}{\kappa}\left[4\pi N_x + x(\sigma)\right] ,$$

  where N_x and N_y are correpsonding winding numbers.
- We impose trivial winding conditions.

Nonassociativity

- Jacobi identity

$$
\begin{aligned}
& \{y_1(\sigma_1), y_2(\sigma_2), y_3(\sigma_3)\} \cong \\
- & \frac{2H}{\kappa^2}\left[\theta(\sigma_1-\sigma_2)\theta(\sigma_2-\sigma_3)+\theta(\sigma_2-\sigma_1)\theta(\sigma_1-\sigma_3)\right. \\
+ & \left.\theta(\sigma_1-\sigma_3)\theta(\sigma_3-\sigma_2)\right] .
\end{aligned}
$$

- For special choice

$$
\{y_1(\sigma+2\pi), y_2(\sigma), y_3(\sigma)\} \cong \frac{2H}{\kappa^2} .
$$

T-dualization along z

- T-dual background fields

$$
{}_zB_{\mu\nu} = \begin{pmatrix} 0 & H\Delta V & 0 \\ -H\Delta V & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad {}_zG_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

- T-dual transformation laws

$$
\partial_\pm z \cong \pm\partial_\pm y_3 \mp H(x\partial_\pm y - y\partial_\pm x),
$$

or in the canonical form

$$
y_3' \cong \frac{1}{\kappa}\pi_z + H(xy' - yx'),
$$

T-dualization along z

- Obtained theory is **nonlocal** (presence of ΔV) - R flux theory.
- Using standard Poisson algebra we can easily come to the fact that z T-dualized theory is **commutative**, and consequently, **associative**.

T-dualization along y

- T-dual background fields

$$
{}_{zy}B_{\mu\nu} = 0\,, \quad {}_{zy}G_{\mu\nu} = \begin{pmatrix} 1 & -2H\Delta V & 0 \\ -2H\Delta V & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

- T-dual transformation law in canonical form

$$
y_2' \cong \frac{1}{\kappa}\pi_y\,.
$$

- Theory is **nonlocal, noncommutative and associative.**

$$
\{y_2(\sigma), y_3(\bar{\sigma})\} \cong -\frac{H}{\kappa}\left[2x(\sigma) - x(\bar{\sigma})\right]\theta(\sigma - \bar{\sigma})\,.
$$

T-dualization along x - full T-dualized theory

- T-dual background fields are of the same form as in the case of $x \to y \to z$ T-dualization chain.
- Noncommutativity relations

$$\{y_1(\sigma), y_3(\bar{\sigma})\} \cong \frac{H}{\kappa}\left[2y(\sigma) - y(\bar{\sigma})\right]\theta(\sigma - \bar{\sigma})\,,$$

$$\{y_2(\sigma), y_3(\bar{\sigma})\} \cong -\frac{H}{\kappa}\left[2x(\sigma) - x(\bar{\sigma})\right]\theta(\sigma - \bar{\sigma})\,.$$

- The same relations as in the case of chain $x \to y \to z$ up to $H \to -H$. Consequently, nonassociativity relation is the same up to the $-$ sign.

Concluding remarks

- We used 3D torus with H-flux in the approximation linear in H to examine dependence of the form of T-dual theory on the direction od T-dualization.
- The full T-dualized theory is not affected by the direction of T-dualization.
- Case $z \to y \to x$ - all three theories are nonlocal (R-flux), but first is commutative, the second one noncommutative and associative, while the full T-dualized theory is noncommutative and nonassociative.
- Case $x \to y \to z$ - first one is geometrical and commuattive, the second one is just locally well defined but still commutative, while the third one is noncommutative and nonassociative.

Literature

1. Bojan Nikolić and Danijel Obrić, Fortschritte der Physik (2018) 1800009, arXiv:1801.08772.
2. Bojan Nikolić and Danijel Obrić, JHEP 03 (2019) 136, arXiv:1901.01040.