Non-Riemannian Volume Elements Dynamically Generate Inflation

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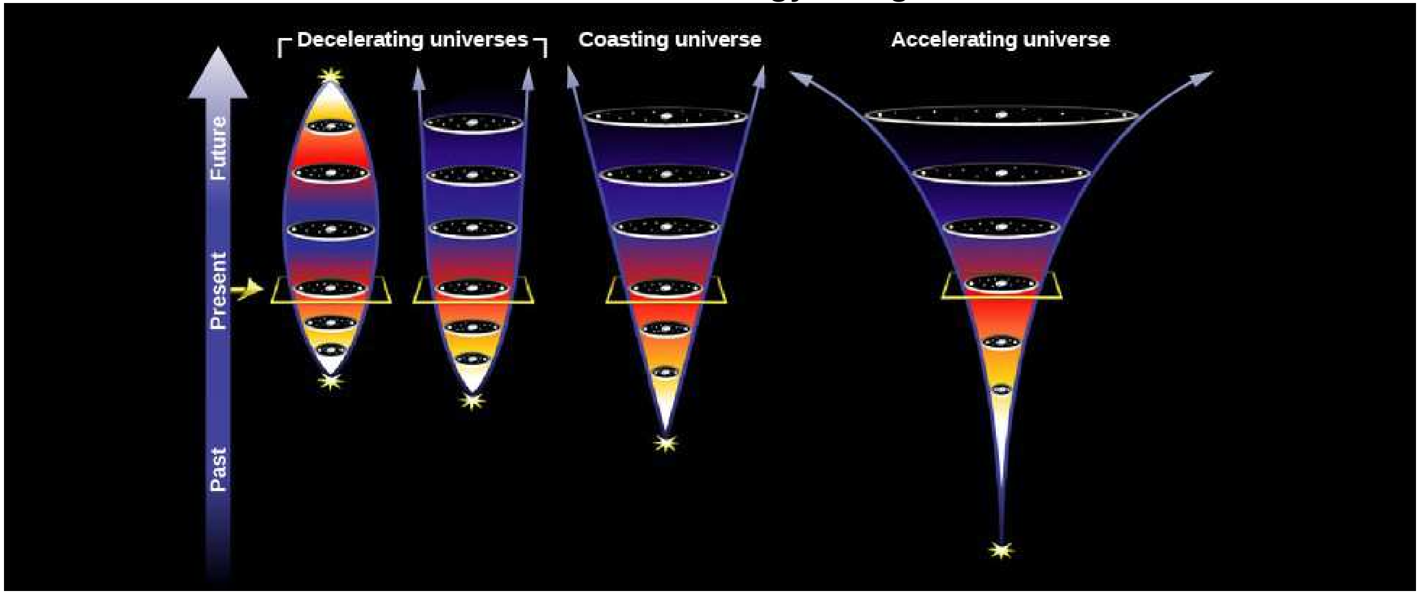


Based on:

- D. Benisty, E. Guendelman, E.Nissimov and S. Pacheva, *arxiv:1906.06691* ; *arxiv:1907.07625*.

Objectives: (a) Formulation of a plausible inflationary model entirely in terms of a pure modified gravity without any *a priori* matter couplings within the formalism of non-Riemannian volume elements;

(b) The non-Riemannian volume elements *dynamically* create in the physical Einstein frame a canonical scalar matter field and produce a non-trivial inflationary potential with a large flat region and a low-lying stable minimum corresponding to the late universe stage;

(c) The model predicts scalar power spectral index and tensor to scalar ratio in accordance with the available data.

Introduction - Overview of Talk

Cosmological Inflation: Explaining the “puzzles” of Big-Bang cosmology (horizon problem, flatness problem, magnetic monople problem, etc.); important framework for treatment of primordial density perturbations, CMB.

Early successful cosmological model: original Starobinsky model based on extended $f(R) = R + R^2$-gravity.

Modified (Extended) Gravity Theories:

Main motivation – to overcome the limitations of standard Einstein’s general relativity: cosmology (problems of dark energy and dark matter), quantum field theory in curved spacetime (renormalization in higher loops), string theory (low-energy effective field theories).

Inflationary Models from Modified Gravity:

Various classes based on: $f(R)$-gravity; scalar-tensor gravity; Gauss-Bonnet gravity, non-local gravity, ... Specific Class – based on the formalism of **non-Riemannian volume elements**, in particular, whithout involving *a priori* **any matter fields**.

Brief Reminder on Non-Riemannian Volume-Forms (Volume Elements)

Essence of the formalism:

In integrals over differentiable manifolds (not necessarily Riemannian one, so *no* metric is needed) volume-forms are given by nonsingular maximal rank differential forms ω:

$$\int_{\mathcal{M}} \omega(\ldots) = \int_{\mathcal{M}} dx^D \, \Omega(\ldots) \quad , \quad \omega = \frac{1}{D!} \omega_{\mu_1 \ldots \mu_D} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_D} \; , \quad (1)$$

where $\omega_{\mu_1 \ldots \mu_D} = -\varepsilon_{\mu_1 \ldots \mu_D} \Omega$; Ω – volume element density.

In Riemannian D-dimensional spacetime manifolds a standard generally-covariant volume-form is defined through the “D-bein” (frame-bundle) canonical one-forms $e^A = e^A_\mu dx^\mu$ ($A = 0, \ldots, D-1$):

$$\omega = e^0 \wedge \ldots \wedge e^{D-1} = \det \| e^A_\mu \| \, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_D} \; , \quad (2)$$

where the standard Riemannian volume element density $\Omega = \det \| e^A_\mu \| = \sqrt{- \det \| g_{\mu\nu} \|}$.

Brief Reminder on Non-Riemannian Volume-Forms (Volume Elements)

To construct modified gravitational theories as alternatives to ordinary standard theories in Einstein's general relativity, instead of $\sqrt{-g}$ we can employ **one or more alternative *non-Riemannian* volume element(s)** as in (1) given by non-singular *exact* D-forms $\omega = dA$, where: $A = \frac{1}{(D-1)!} A_{\mu_1 \ldots \mu_{D-1}} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu-1}$ and the corresponding volume element density reads:

$$\Omega \equiv \Phi(A) = \frac{1}{(D-1)!} \varepsilon^{\mu_1 \ldots \mu_D} \partial_{\mu_1} A_{\mu_2 \ldots \mu_D} \ . \tag{3}$$

Thus, non-Riemannian volume element densities $\Phi(A)$ are defined in terms of the (scalar density of the) dual field-strength of auxiliary rank $D-1$ tensor gauge fields $A_{\mu_1 \ldots \mu_{D-1}}$.

Remark: In the first-order (Palatini) formalism ($g_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$ *a priori* independent), the auxiliary tensor gauge fields $A_{\mu_1 \ldots \mu_{D-1}}$ turn out to be (almost) pure-gauge – *no propagating field degrees of freedom* except few integration constants.

Brief Reminder on Non-Riemannian Volume-Forms (Volume Elements)

However, in the second-order (metric) formalism (where $\Gamma^{\lambda}_{\mu\nu}$ is the usual Levi-Civita connection of the metric $g_{\mu\nu}$) the first non-Riemannian volume form $\Phi(A)$ is not any more pure-gauge.

The reason is that in the action $S = \int d^4x \Phi(A) R + \ldots$, the scalar curvature R (in the metric formalism) containes **second-order** (time) derivatives (they amount to a total derivative in the ordinary case $S = \int d^4x \sqrt{-g} R + \ldots$).

So defining $\chi_1 \equiv \Phi(A)/\sqrt{-g}$ – it becomes physical degree of freedom as seen from the eqs. of motion:

$$R_{\mu\nu} + \frac{1}{\chi_1}\left(g_{\mu\nu}\Box\chi_1 - \nabla_\mu\nabla_\nu\chi_1\right) + \ldots = 0 \qquad (4)$$

Pure Gravity with Non-Riemannian Volume Elements

Simple modified gravity model without any matter fields (using "Planck units" $16\pi G_N = 1$):

$$S = \int d^4x \Big\{ \Phi_1(A) \Big[R - 2\Lambda_0 \frac{\Phi_1(A)}{\sqrt{-g}} \Big] + \Phi_2(B) \frac{\Phi_0(C)}{\sqrt{-g}} \Big\} \; , \tag{5}$$

R is the scalar curvature in the metric formalism and:

$$\Phi_1(A) \equiv \frac{1}{3!} \varepsilon^{\mu\nu\kappa\lambda} \partial_\mu A_{\nu\kappa\lambda} \; , \; \Phi_2(B) \equiv \frac{1}{3!} \varepsilon^{\mu\nu\kappa\lambda} \partial_\mu B_{\nu\kappa\lambda} \; ,$$
$$\Phi_0(C) \equiv \frac{1}{3!} \varepsilon^{\mu\nu\kappa\lambda} \partial_\mu C_{\nu\kappa\lambda} \; . \tag{6}$$

Λ_0 is dimensionful parameter (will play the role of inflationary scale).

The form of the action (5) is uniquely specified by the requirement about global Weyl-scale invariance under:

$$g_{\mu\nu} \to \lambda g_{\mu\nu} \; , \; A_{\mu\nu\kappa} \to \lambda A_{\mu\nu\kappa} \; , \; B_{\mu\nu\kappa} \to \lambda^2 B_{\mu\nu\kappa} \; , \; C_{\mu\nu\kappa} \to C_{\mu\nu\kappa} \; . \tag{7}$$

where $\lambda =$ const.

Pure Gravity with Non-Riemannian Volume Elements

The eqs. motion from (5) w.r.t. the auxiliary gauge fields $A_{\mu\nu\lambda}$, $B_{\mu\nu\lambda}$, $C_{\mu\nu\lambda}$ yield, respectively:

$$R - 4\Lambda_0 \frac{\Phi_1(A)}{\sqrt{-g}} = -M_1 \equiv \text{const} \;, \tag{8}$$

$$\frac{\Phi_0(C)}{\sqrt{-g}} = -M_2 \equiv \text{const} \;\; , \;\; \frac{\Phi_2(B)}{\sqrt{-g}} = \chi_2 \equiv \text{const} \; . \tag{9}$$

Here M_1, M_2 and χ_2 are free integration constants; M_1, M_2 indicate spontaneous breaking of global Weyl symmetry (7).

The eqs. motion w.r.t. $g_{\mu\nu}$ from (5) read:

$$R_{\mu\nu} - \Lambda_0 \chi_1 \, g_{\mu\nu} + \frac{1}{\chi_1}\Big(g_{\mu\nu}\Box\chi_1 - \nabla_\mu\nabla_\nu\chi_1\Big) - \frac{\chi_2 M_2}{\chi_1} g_{\mu\nu} = 0 \;, \tag{10}$$

with $\chi_1 \equiv \Phi(A)/\sqrt{-g}$. Taking the trace of (10):

$$3\frac{\Box\chi_1}{\chi_1} - \frac{4\chi_2 M_2}{\chi_1} - M_1 = 0 \; . \tag{11}$$

Einstein Frame and Effective Scalar Potential

We now transform Eqs.(10) and (11) via conformal transformation $\bar{g}_{\mu\nu} = \chi_1 g_{\mu\nu}$ – transformed equations acquire the standard form of Einstein equations w.r.t. the new "Einstein-frame" metric $\bar{g}_{\mu\nu}$:

$$R_{\mu\nu}(\bar{g}) - \frac{1}{2}\bar{g}_{\mu\nu}R(\bar{g}) = \frac{1}{2}\Big[\partial_\mu u \partial_\nu u - \bar{g}_{\mu\nu}\big(\frac{1}{2}\bar{g}^{\kappa\lambda}\partial_\kappa u \partial_\lambda u + U_{\text{eff}}(u)\big)\Big] \; , \quad (12)$$

$$\bar{\Box} u + \frac{\partial U_{\text{eff}}}{\partial u} = 0 \; , \quad (13)$$

where we have redefined $\Phi_1(A)/\sqrt{-g} \equiv \chi_1 = \exp\big(u/\sqrt{3}\big)$ in order to obtain a canonically normalized kinetic term for the scalar field u, and where we have a **dynamically generated effective scalar potential**:

$$U_{\text{eff}}(u) = 2\Lambda_0 - M_1 \exp\big(-\frac{u}{\sqrt{3}}\big) + \chi_2 M_2 \exp\big(-2\frac{u}{\sqrt{3}}\big) \; . \quad (14)$$

U_{eff} (14) is a generalization of the classic **Starobinsky potential** – it is a special case of (14) for $\Lambda_0 = \frac{1}{4}M_1 = \frac{1}{2}\chi_2 M_2$.

Einstein Frame and Effective Potential

Accordingly, the corresponding Einstein-frame action reads:

$$S_{\rm EF} = \int d^4x \sqrt{-\bar{g}} \Big[R(\bar{g}) - \frac{1}{2} \bar{g}^{\mu\nu} \partial_\mu u \partial_\nu u - U_{\rm eff}(u) \Big] \; . \qquad (15)$$

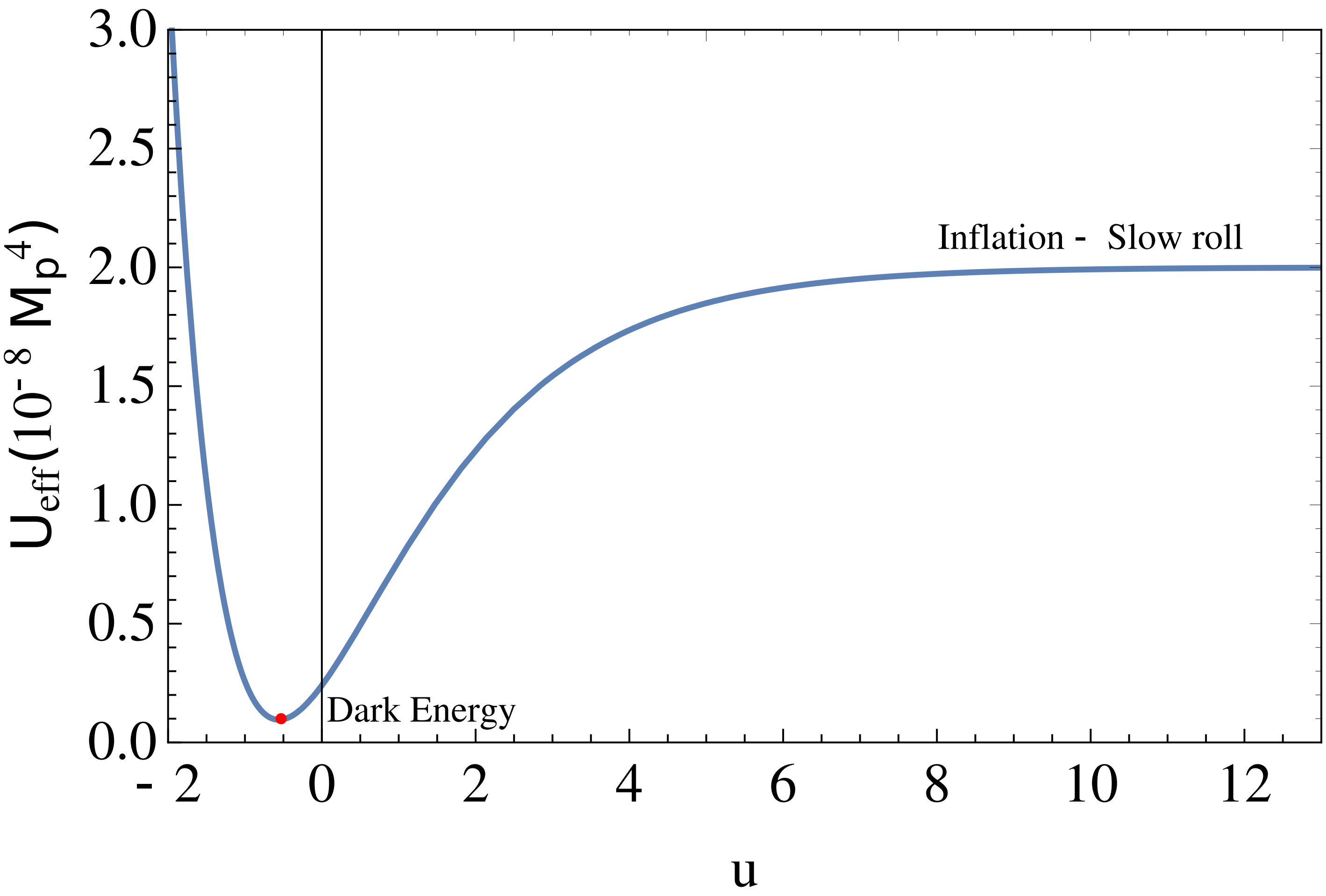
$$U_{\rm eff}(u) = 2\Lambda_0 - M_1 \exp\big(-\frac{u}{\sqrt{3}}\big) + \chi_2 M_2 \exp\big(-2\frac{u}{\sqrt{3}}\big) \; .$$

The Einstein-frame action is entirely dynamically generated:

(a) The canonical scalar field u is dynamically created from the ratio of volume-element densities $\Phi(A)/\sqrt{-g}$;

(b) The effective potential $U_{\rm eff}(u)$ is dynamically generated due to the appearance of the free integration constants $M_{1,2}, \chi_2$ as a result of the constrained dynamics of the auxiliary gauge fields $A_{\mu\nu\lambda}, B_{\mu\nu\lambda}, C_{\mu\nu\lambda}$ – constituents of the non-Riemannian volume element densities $\Phi(A), \Phi(B), \Phi(C)$.

Einstein Frame and Effective Potential



Qualitative shape of the dynamically generated effective scalar potential $U_{\rm eff}$ (14) as function of u. The unit for u is $M_{Planck}/\sqrt{2}$.

Einstein Frame and Effective Potential

$U_{\rm eff}(u)$ has two main features relevant for cosmological applications.

First, $U_{\rm eff}(u)$ (14) possesses a flat region for large positive u and, second, it has a stable minimum for a small finite value $u = u_*$:

- (i) $U_{\rm eff}(u) \simeq 2\Lambda_0$ for large u;
- (ii) $\frac{\partial U_{\rm eff}}{\partial u} = 0$ for $u \equiv u_*$ where:

$$\exp\Big(-\frac{u_*}{\sqrt{3}}\Big) = \frac{M_1}{2\chi_2 M_2} \quad , \quad \frac{\partial^2 U_{\rm eff}}{\partial u^2}\Big|_{u=u_*} = \frac{M_1^2}{6\chi_2 M_2} > 0 \; . \qquad (16)$$

The flat region of $U_{\rm eff}(u)$ for large positive u correspond to "early" universe' inflationary evolution with energy scale $2\Lambda_0$. On the other hand, the region around the stable minimum at $u = u_*$ (16) correspond to "late" universe' evolution where:

$$U_{\rm eff}(u_*) = 2\Lambda_0 - \frac{M_1^2}{4\chi_2 M_2} \equiv 2\Lambda_{\rm DE} \qquad (17)$$

is the dark energy density value.

Evolution of the Homogeneous Solution

Consider reduction of the Einstein-frame action (15) to the Friedmann-Lemaitre-Robertson-Walker (FLRW) setting with metric $ds^2 = -N^2 dt^2 + a(t)^2 d\vec{x}^2$, and with $u = u(t)$.

Will study the evolution of $u = u(t)$ and $a = a(t)$ using the method of autonomous dynamical systems.

FLRW-reduced action:

$$S_{\rm FLRW} = \int d^4x \Big[-6 \frac{a\,\dot{a}^2}{N} + N a^3 \Big(\frac{1}{2}\,\dot{u}^2 + M_1 e^{-u/\sqrt{3}} - M_2 \chi_2 e^{-2u/\sqrt{3}} - 2\Lambda_0 \Big) \Big] \tag{18}$$

The pertinent Friedmann and u-field equations:

$$H^2 = \frac{1}{6}\rho \;\; , \;\; \rho = \frac{1}{2}\,\dot{u}^2 + U_{\rm eff}(u) \; , \tag{19}$$

$$\dot{H} = -\frac{1}{4}(\rho + p) \;\; , \;\; p = \frac{1}{2}\,\dot{u}^2 - U_{\rm eff}(u) \; , \tag{20}$$

$$\ddot{u} + 3H\,\dot{u} + \frac{\partial U_{\rm eff}}{\partial u} = 0 \; . \tag{21}$$

Rewrite Eqs.(19)-(21) in terms of dimensionless variables:

$$x := \frac{\dot{u}}{\sqrt{12}H}, \quad y := \frac{\sqrt{U_{\rm eff}(u) - 2\Lambda_{\rm DE}}}{\sqrt{6}H}, \quad z := \frac{\sqrt{\Lambda_{\rm DE}}}{\sqrt{3}H} \;, \tag{22}$$

with $L_{\rm DE} = \Lambda_0 - \frac{M_1^2}{8\chi_2 M_2}$ as in (17).

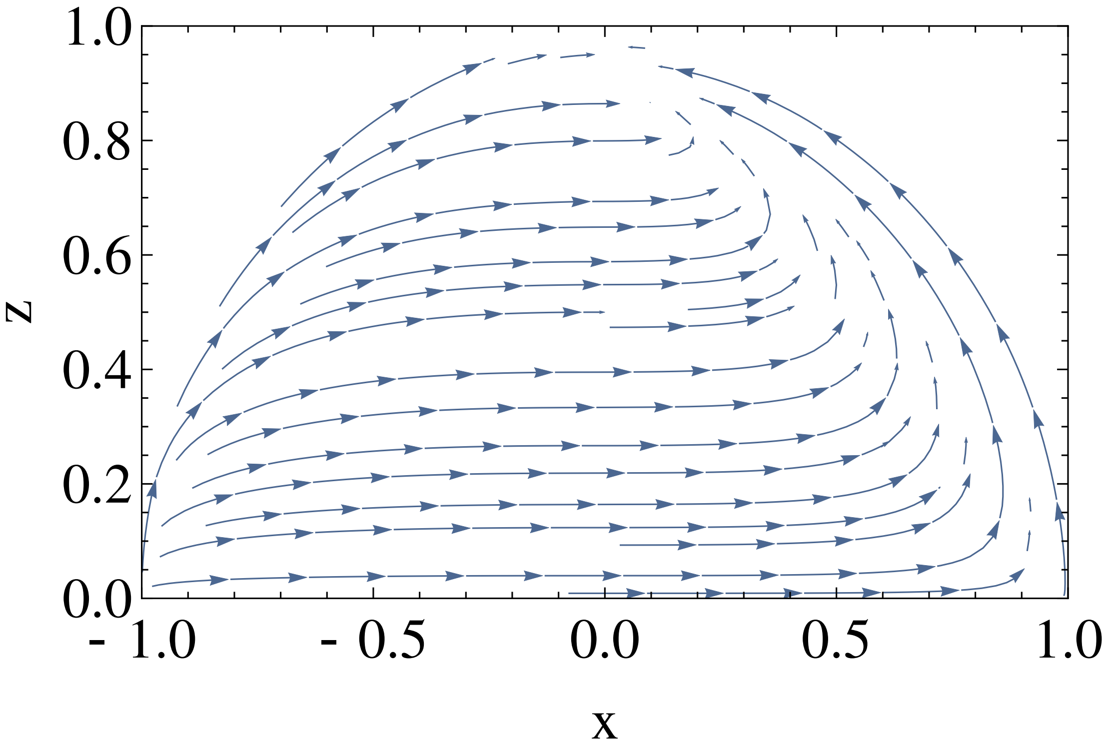
The first Friedman Eq.(19) yields an algebraic constraint $x^2 + y^2 + z^2 = 1$, so that the autonomous dynamical system w.r.t. (x, z) reads:

$$x' = \frac{\sqrt{3}}{2\Lambda_{DE}} z^2 \left[-M_1 \xi(x,z) + 2M_2\chi_2\xi^2(x,z)\right] - 3x(1-x^2) \;,$$
$$z' = 3zx^2 \quad , \tag{23}$$

where the primes denote derivative w.r.t. the parameter $\mathcal{N} = \log a$, and the function $\xi(x,z)$ is defined as:

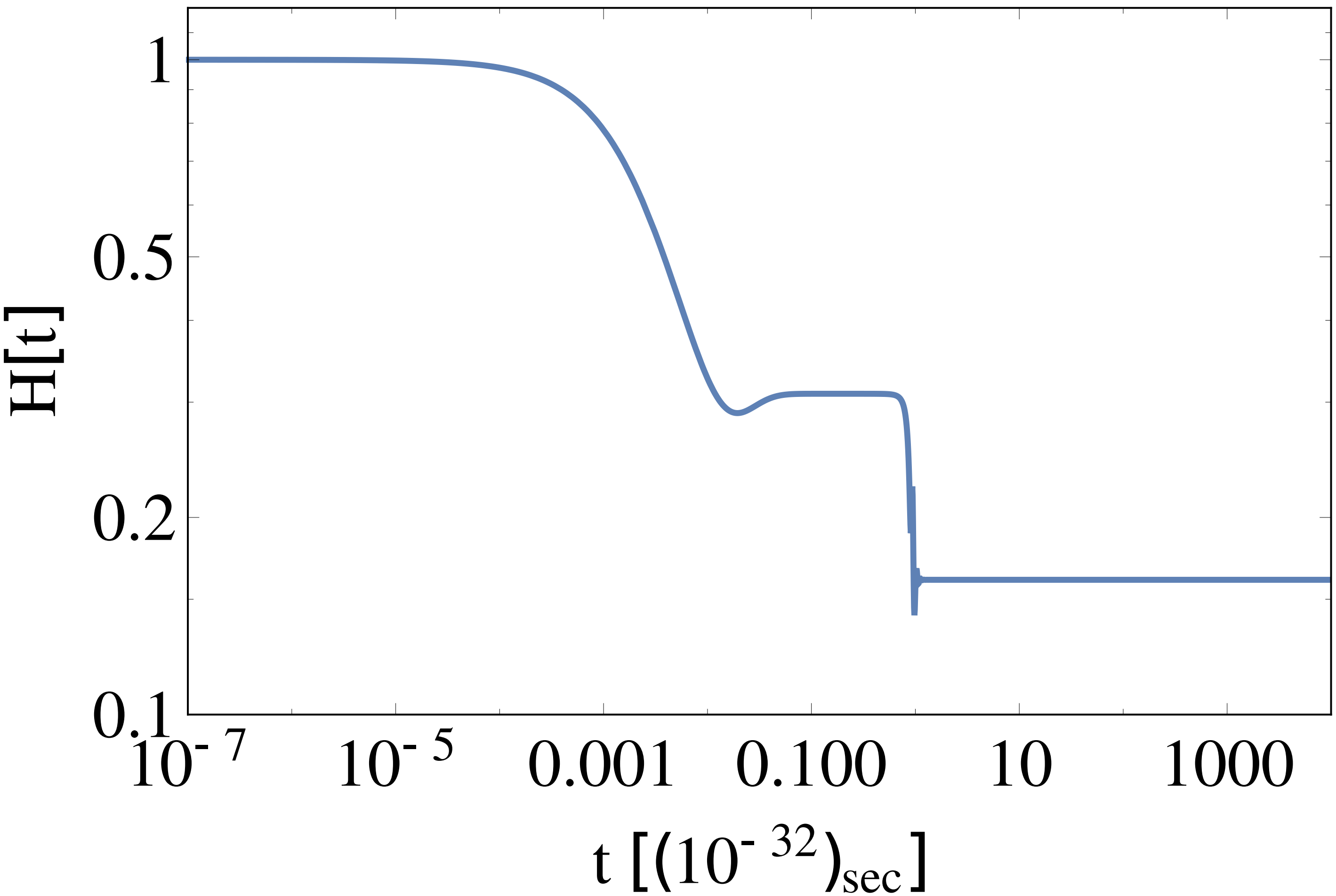
$$\xi(x,z) = \frac{M_1}{2\chi_2 M_2}\left[1 - \sqrt{\frac{8\Lambda_0 M_2 \chi_2}{M_1^2}\,\frac{1-x^2-z^2}{z^2}}\right] \,. \tag{24}$$

Phase space portrait of the autonomous system (23) – 2 crit. points:

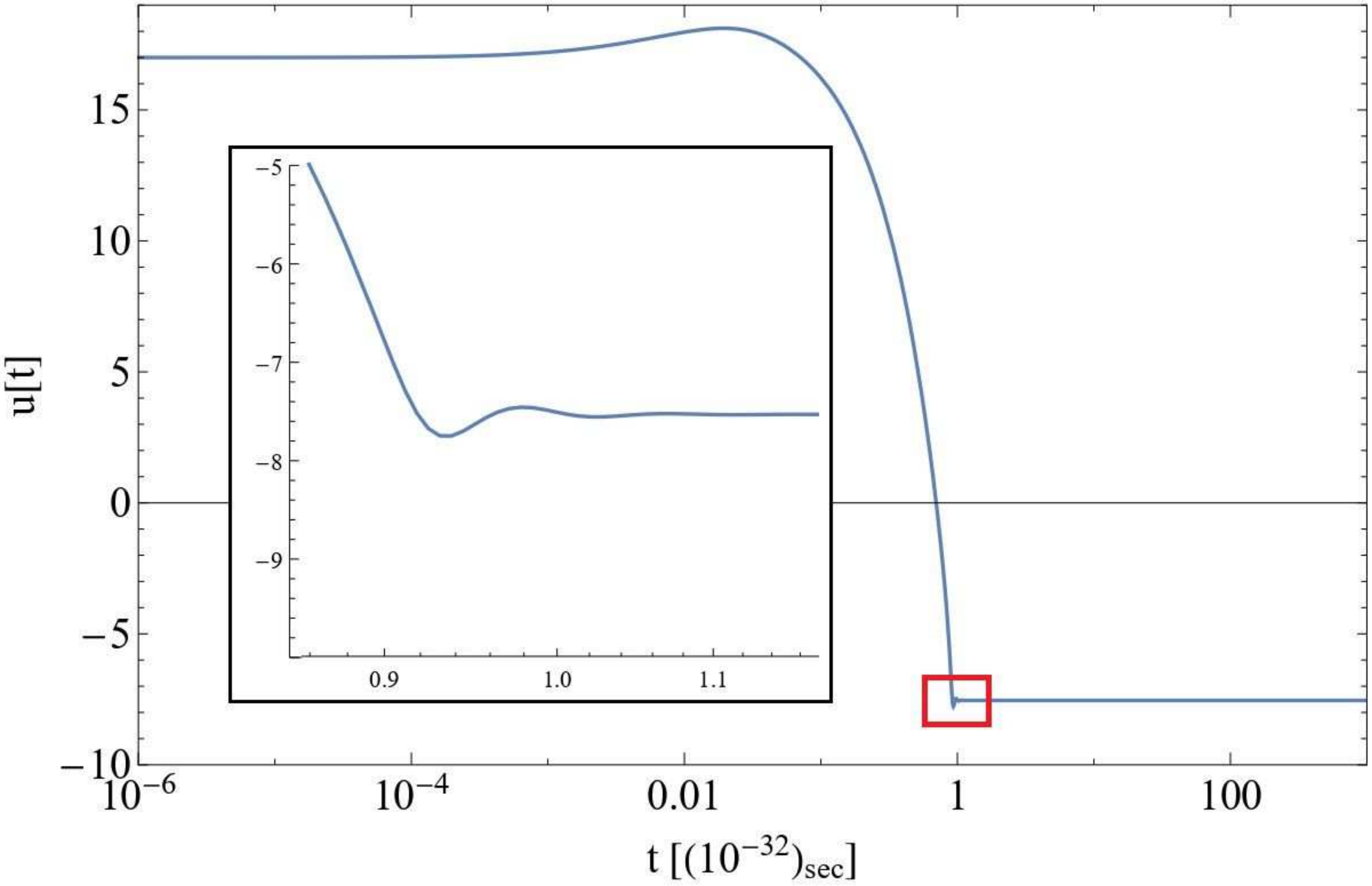


(a) Stable point $A\,(x=0, z=1)$ corresponding to "late" universe de Sitter behavior with CC $\Lambda_{\rm DE}$ (17).

(b) Unstable point $B\,\left(x=0, z=\sqrt{\Lambda_{\rm DE}/\Lambda_0}\right)$ corresponding to beginning of evolution in the "early" universe at large u. If the evolution starts at any point close to B, then the dynamics drives the system away from B towards the stable point A at late times.



Numerical example of the solution for the Hubble parameter $H(t)$ vs. time. Initially for short times the inflationary Hubble parameter is large and afterwards approaches its cosmological late time value.



Numerical example of the solution for the scalar field $u(t)$ vs. time. The unit for u is $M_{Planck}/\sqrt{2}$. The blown-up rectangle depicts the oscillations of $u(t)$ around the minimum of $U_{\rm eff}$ (14).

Perturbations and Observables

In order to check the viability of the model we investigate the perturbations of the above background evolution, in particular focusing on the inflationary observables such as the scalar spectral index n_s and the tensor-to-scalar ratio r. As usual, we introduce the Hubble slow-roll parameters, which in our case using the potential $U_{\rm eff}(u)$ (14) read:

$$\epsilon = \Big(\frac{U'_{\text{eff}}(u)}{U_{\text{eff}}(u)}\Big)^2 = \frac{4\zeta^2}{3} \frac{\big(1/2-\zeta\big)^2}{\big[\big(1/2-\zeta\big)^2 + \delta/4\big]^2} \, , \tag{25}$$

$$|\eta| = 2\Big|\frac{U''_{\text{eff}}(u)}{U_{\text{eff}}(u)}\Big| = \frac{2\zeta}{3} \frac{\big(1-4\zeta\big)}{\big[\big(1/2-\zeta\big)^2 + \delta/4\big]} \, , \tag{26}$$

where:

$$\zeta \equiv \frac{M_2\chi_2}{M_1}\, e^{-u/\sqrt{3}} \quad , \quad \delta \equiv \frac{8M_2\chi_2}{M_1^2}\Lambda_{\rm DE} \, , \tag{27}$$

with $\Lambda_{\rm DE}$ – the dark energy density (17), and therefore δ very small.

Perturbations and Observables

Inflation ends when $\epsilon(u_f) = 1$ for some $u = u_f$ where ($\zeta_f \equiv \frac{M_2\chi_2}{M_1} e^{-u_f/\sqrt{3}}$):

$$\zeta_f = \frac{1}{2\big(1+2/\sqrt{3}\big)}\Big[1 + \frac{1}{\sqrt{3}} - \sqrt{1/3 - \big(1+2/\sqrt{3}\big)\delta}\Big]$$

$$\simeq \frac{1}{2\big(1+2/\sqrt{3}\big)} \; . \qquad (28)$$

For the number of e-foldings $\mathcal{N} = \frac{1}{2}\int_{u_i}^{u_f} du\, U_{\rm eff}/U'_{\rm eff}$ we obtain:

$$\mathcal{N} = \frac{3}{8}(1+\delta)\Big(1/\zeta_i - 1/\zeta_f\Big)$$

$$-\frac{3}{4}(1-\delta)\log\frac{\zeta_f}{\zeta_i} + \frac{3}{4}\delta\,\log\Big(\frac{1-2\zeta_i}{1-2\zeta_f}\Big) \; , \qquad (29)$$

where $\zeta_i \equiv \frac{M_2\chi_2}{M_1} e^{-u_i/\sqrt{3}}$ and $u = u_i$ is very large corresponding to the start of the inflation.

Perturbations and Observables

Ignoring δ we have for $\mathcal{N}$ approximately:

$$\mathcal{N} \simeq \frac{3M_1}{8M_2\chi_2} e^{u_i/\sqrt{3}} - \frac{\sqrt{3}}{4} u_i - \frac{3}{4}\left(1+2/\sqrt{3}\right) + \frac{3}{4}\log\Big(2\left(1+2/\sqrt{3}\right)\Big) \;. \quad (30)$$

Using the slow-roll parameters, one can calculate the values of the scalar spectral index n_s and the tensor-to-scalar ratio r, respectively, as functions of $\mathcal{N}$:

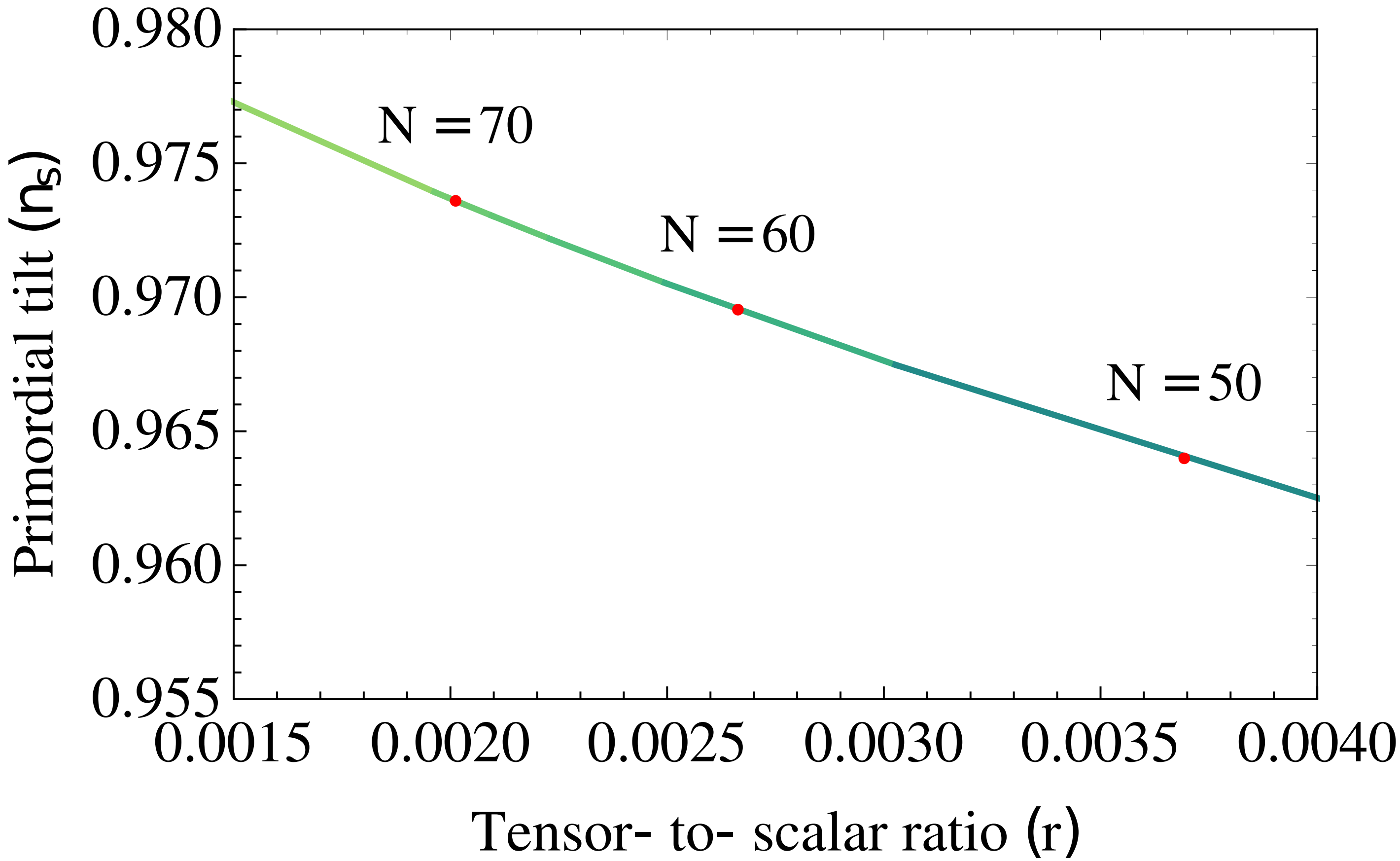
$$r \approx 16\,\epsilon\big(u_i(\mathcal{N})\big) \quad , \quad n_s \approx 1 - 6\,\epsilon\big(u_i(\mathcal{N})\big) + 2\eta\,\big(u_i(\mathcal{N})\big) \;, \quad (31)$$

where $u_i(\mathcal{N})$ is the solution of the transcedental Eq.(30) for u_i as a function of $\mathcal{N}$. From (31), (30), (25), (26) we find:

$$r \simeq \frac{12}{\left[\mathcal{N} + \frac{\sqrt{3}}{4} u_i(\mathcal{N}) + c_0\right]^2} \quad , \quad c_0 \equiv \frac{\sqrt{3}}{2} - \frac{3}{4}\log\Big(2\left(1+2/\sqrt{3}\right)\Big) \;, \quad (32)$$

and $n_s \simeq 1 - \frac{r}{4} - \sqrt{\frac{r}{3}}$.

Perturbations and Observables



The predicted values of the r and n_s for different e-foldings. The different values of the r and n_s are compatible with the PLANCK observational data ($0.95 < n_s < 0.97$, $r < 0.064$).

Viable example – for $\mathcal{N} = 60$ we obtain: $n_s \approx 0.969$, $r \approx 0.002$.

Conclusions

- We proposed a very simple gravity model without any initial matter fields in terms of several alternative non-Riemannian spacetime volume elements within the second order (metric) formalism.
- We show how the non-Riemannian volume-elements, when passing to the physical Einstein frame, create a canonical scalar field and produce dynamically a non-trivial inflationary-type potential for the latter with a large flat region and a stable low-lying minimum.
- We study the evolution of the cosmological solutions from the point of view of the theory of dynamical systems. Our model predicts scalar spectral index $n_s \approx 0.969$ and tensor-to-scalar ratio $r \approx 0.002$ for 60 e-folds, which is in accordance with the observational data.

Conclusions

- A natural next step is to consider two-field inflation by adding a new scalar field φ with non-trivial potentials in the starting modified gravity action (5) built in terms of several non-Riemannian volume elements and subject to preserving the requirement of global Weyl-scale invariance (7).
- In this case the non-Riemannian volume elements will again generate a second scalar field u and create dynamically a non-trivial two-field scalar potential with a very specific geometry of the field space of φ, u.

It will be interesting to see how the latter dynamically generated two-field inflationary model would conform to the observational data.

THANK YOU FOR YOUR PATIENCE