Digital quantum geometries

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based on the joint work with Shahn Majid

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Introduction

Motivation from Quantum Gravity

Continuum differential geometry cannot be the geometry when both quantum and gravitational effects are present

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Continuum differential geometry cannot be the geometry when both quantum and gravitational effects are present

One of the possibilities is to consider

NONCOMMUTATIVE GEOMETRY (NCG)

where the idea is to "algebralize" geometric notions and then generalize them to noncommutative algebras

- Noncommutative Geometry $\leftrightarrow$ Quantum geometry:

  On a curved space one must use the methods of Riemannian geometry but in their quantum version.

- The formalism of noncommutative differential geometry does not require functions and differentials to commute, so is more general even when the algebra is classical.

Plan of the talk:

1. Quantum Riemannian Geometry ingredients
2. Digital - what & why
3. Digital quantum geometries in $n \leq 3$
4. Conclusions

Differential Geometry vs NC Differential Geometry

M - manifold and
$C^\infty(M)$ - functions on a manifold

$\rightarrow$ 'coordinate algebra' A

and

Ω^1 space of 1-forms, e.g. differentials:

$$\mathrm{d}f = \sum_i \frac{\partial f}{\partial x^\mu} \mathrm{d}x^\mu$$

$$f \mathrm{d}g = (\mathrm{d}g) f$$

$\rightarrow$ noncommutative differential structure:
differential bimodule (Ω^1, d) of 1-forms with d - obeying the Leibniz rule and
$\rightarrow f \mathrm{d}g \neq (\mathrm{d}g) f$

Bimodule - to associatively multiply such 1-forms by elements of A from the left and the right.

Quantum Riemannian Geometry

Ingredients of noncommutative Riemannian geometry as quantum geometry:

- quantum differentials
- quantum metrics
- quantum-Levi Civita connections
- quantum curvature
- quantum Ricci and Einstein tensors

Quantum differentials

Differential calculus on an algebra A

- A is a 'coordinate' algebra (noncommutative or commutative) over any field k.

Definition

A first order differential calculus (Ω^1, d) over A means:

1. Ω^1 is an A-bimodule
2. A linear map $\mathrm{d} : A \to \Omega^1$ such that

$$\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b \quad , \forall a, b \in A$$

3. $\Omega^1 = span\{adb\}$
4. (optional) $\ker d = k.1$ - connectedness condition

Differential graded algebra -DGA

Definition

DGA on an algebra A is:

1. A graded algebra $\Omega = \oplus_{n \geq 0} \Omega^n$, $\Omega^0 = A$
2. $\mathrm{d} : \Omega^n \to \Omega^{n+1}$, s.t. $\mathrm{d}^2 = 0$ and
$$\mathrm{d}(\omega\rho) = (\mathrm{d}\omega) \wedge \rho + (-1)^n \omega \wedge \mathrm{d}\rho$$
$\forall \omega, \rho \in \Omega, \quad \omega \in \Omega^n.$
3. $A, \mathrm{d}A$ generate Ω
(optional surjectivity condition - if it holds we say it is an **exterior algebra** on A)

Quantum metrics

When working with algebraic differential forms by **metric** we mean an element

$$g \in \Omega^1 \otimes_A \Omega^1$$

which is:

- 'quantum symmetric': $\wedge(g) = 0$,
- invertible
  in the sense that there exists $(\ \ ,\ \ ) : \Omega^1 \otimes_A \Omega^1 \to A$
  $$((\omega,\ \ ) \otimes id)g = \omega = (id \otimes (\ \ , \omega))g \qquad \forall \omega \in \Omega^1$$

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- **central in the 'coordinate algebra'** $A \ni x^\mu$:
  $$[g, x^\mu] = 0$$

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  $$\underline{\dim} = (\ ,\ )(g) \in k.$$

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The general form of the quantum metric:

$$g = g_{\mu\nu} dx^\mu \otimes_A dx^\nu$$

Quantum connections

[Quillen, Karoubi, Michor, Mourad, Dubois-Violette, . . .]

- Bimodule connection: $\nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$,
  $\sigma \quad : \quad \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$,
  for $a \in A, \omega \in \Omega^1$

$$\nabla(a\omega) = a\nabla\omega + \mathrm{d}a \otimes \omega$$

$$\nabla(\omega a) = (\nabla\omega)a + \sigma(\omega \otimes \mathrm{d}a)$$

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- Such connections extend to tensor products:

$$\nabla(\omega\otimes\eta) = (\nabla\omega)\otimes\eta + (\sigma\otimes id)(\omega\otimes\nabla\eta), \qquad \omega\otimes\eta \in \Omega^1\otimes_A\Omega^1$$

Metric compatibility, torsion and curvature

Metric compatible connection:

$$\nabla(g) = 0$$

Torsion of a connection on Ω^1 is

$$T_{\nabla}\omega = \wedge\nabla\omega - \mathrm{d}\omega \quad : \qquad T_{\nabla} : \Omega^1 \to \Omega^2$$

We define a **quantum Levi-Civita connection (QLC connection)** as metric compatible and torsion free connection.

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Curvature:

$$R_\nabla \omega = (\mathrm{d} \otimes id - \wedge(id \otimes \nabla))\nabla \omega \qquad R_\nabla : \quad \Omega^1 \to \Omega^2 \otimes_A \Omega^1$$

Ricci & Einstein tensors

- Ricci tensor:

$$\mathrm{Ricci} = ((\ ,\ ) \otimes \mathrm{id})(\mathrm{id} \otimes i \otimes \mathrm{id})R_\nabla$$

  with respect to a 'lifting' bimodule map $i : \Omega^2 \to \Omega^1 \otimes_A \Omega^1$ such that $\wedge \circ i = \mathrm{id}$.
- Then Ricci scalar is $S = (\ ,\ )\mathrm{Ricci}$.

[Beggs,Majid,Class.Quantum.Grav.31(2014)]

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- For Einstein tensor one can consider the usual definition

$$\mathrm{Eins} = \mathrm{Ricci} - \frac{1}{2} Sg$$

  but field independent option would be:

$$\mathrm{Eins} = \mathrm{Ricci} - \alpha Sg, \qquad \alpha \in k$$

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- one could take $\mathrm{Eins} = \mathrm{Ricci} - \frac{1}{\mathrm{dim}}Sg$

[Beggs,Majid,Class.Quantum.Grav.31(2014)]

'Digital'

Recall that the framework works for A over any field k.
Take k as the finite field $\mathbb{F}_2 = \{0, 1\}$.

- The choice of the finite field leads to a new kind of 'discretisation scheme', which adds **'digital'** to quantum geometry.
- A standard technique in physics/engineering is to replace geometric backgrounds by **discrete approximations** such as a lattice or graph, thereby rendering systems more calculable.
- Allows to get a **repertoire of digital quantum geometries** $\Rightarrow$ to test ideas and conjectures in the general theory if we expect them to hold for any field, even if we are mainly interested in the theory over $\mathbb{C}$.

Aim

- to study bimodule quantum Riemannian geometries over the field $\mathbb{F}_2 = \{0, 1\}$ of two elements (**'digital'** quantum geometries)
- to classify all such geometries for coordinate algebras up to dimension $n \leq 3$

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Preview of results:
A rich moduli of examples for $n = 3$, including **9 that are Ricci flat but not flat**
(with commutative coordinate algebras $x^\mu x^\nu = x^\nu x^\mu$,
but with noncommuting differentials $x^\mu dx^\rho \neq dx^\rho x^\mu$,
$x^\mu, x^\nu \in A, dx^\rho \in \Omega^1$).

Digital Quantum Geometry set up

- **'Coordinate algebra' A (unital associative algebra) over** $\mathbb{F}_2$ - **the field of two elements** $0, 1$**.**
- $\{x^\mu\}$ - basis of A where $x^0 = 1$ the unit and $\mu = 0, \cdots, n-1$.
- Structure constants $V^{\mu\nu}{}_\rho \in \mathbb{F}_2$

$$x^\mu x^\nu = V^{\mu\nu}{}_\rho x^\rho.$$

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We have classified all possible such algebras over $\mathbb{F}_2$ up to $n \leq 4$. [S.Majid,A.P.,J.Math.Phys.59 (2018)]

'Coordinate algebras' over $\mathbb{F}_2$ in low dim

$\{x^\mu\}$ is a basis of A where $x^0 = 1$ the unit and $\mu = 0, \cdots, n-1$

For $n = 1$

There is only **one** unital algebra of dimension 1 ($x^0 x^0 = x^0$)

For $n = 2$

There are **3*** inequivalent (commutative) algebras A, B, C:

A: $x^1 x^1 = 0$

B: $x^1 x^1 = x^1$

C: $x^1 x^1 = x^0 + x^1 = 1 + x^1$.

For $n = 3$

There are **6*** inequivalent (commutative) algebras: A, B, C, D, E, F and **one** noncommutative G.

For $n = 4$ There are **16*** inequivalent (commutative) algebras: A - P and 9 noncommutative ones.

* up to isomorphisms

Classification of quantum digital geometries for $n = 3$

- We have considered each of the 6 commutative (A-F) and one noncommutative (G) algebras with two dimensional Ω^1 (**the universal calculus**) and with 1 dimensional Ω^1.

- To keep things simple, for the universal calculus, we considered geometries with basis $\omega^1 = \mathrm{d}x^1, \omega^2 = \mathrm{d}x^2$ for Ω^1 and we take 1 dimensional Ω^2

Digital quantum geometries - one algebra example

- From the 6 algebras (A - F) let's choose algebra D (an example of 3-dimensional unital commutative algebra with the basis $1, x^1, x^2$).
- Relations: $x^1x^1 = x^2, \quad x^2x^2 = x^1, \quad x^1x^2 = x^1 + x^2 = x^2x^1$
- **Universal differential calculus** with relations:

$$\mathrm{d}x^1.x^2 = x^1\mathrm{d}x^2 + \mathrm{d}x^1 + \mathrm{d}x^2, \qquad \mathrm{d}x^2.x^1 = x^2\mathrm{d}x^1 + \mathrm{d}x^1 + \mathrm{d}x^2$$

$$[\mathrm{d}x^1, x^1] = \mathrm{d}x^2, \qquad [\mathrm{d}x^2, x^2] = \mathrm{d}x^1$$

Basis of Ω^1: $\omega^1 = \mathrm{d}x^1, \omega^2 = \mathrm{d}x^2$

- This algebra (D) is isomorphic to $\mathbb{F}_2\mathbb{Z}_3$ the group algebra on the group $\mathbb{Z}_3$ since $z = 1 + x^1$ obeys $(z)^2 = 1 + x^2$ and $(z)^3 = 1$.

Quantum metric on $\mathbb{F}_2\mathbb{Z}_3$

We define a metric as an invertible element of $g \in \Omega^1 \otimes_D \Omega^1$.

$$g = g_{ij}\omega^i \otimes \omega^j = g_{\mu ij}x^\mu \omega^i \otimes \omega^j, \quad g_{ij} \in D, \quad g_{\mu ij} \in \mathbb{F}_2$$

- Quantum metric (central and quantum symm.) on $D = \mathbb{F}_2\mathbb{Z}_3$:

$$g_D = \beta z^2 \omega^1 \otimes \omega^1 + \beta z(\omega^1 \otimes \omega^2 + \omega^2 \otimes \omega^1) + \beta \omega^2 \otimes \omega^2$$

  with β - a functional parameter.

- We take special cases for $\beta = 1, z, z^2$
- For these there are 12 QLC connections (11 of them not flat! $R_\nabla \neq 0$ - purely 'quantum' phenomenon.)

Digital quantum connection and curvature

with the structure constants in $\mathbb{F}_2$:

$$\nabla\omega^i = \Gamma^i{}_{\nu km} x^\nu \omega^k \otimes \omega^m, \quad \sigma\left(\omega^i \otimes \omega^j\right) = \sigma^{ij}{}_{\mu km} x^\mu \omega^k \otimes \omega^m,$$

$\Gamma^i{}_{\nu km},\ \sigma^{ij}{}_{\mu km} \in \mathbb{F}_2$.

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For the curvature $R_\nabla : \Omega^1 \to \Omega^2 \otimes_D \Omega^1$:

$$R_\nabla = (\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla)\nabla$$

$$R_\nabla \omega^i = \rho^i{}_{j\mu} x^\mu \mathrm{Vol} \otimes \omega^j = \rho^i{}_j \mathrm{Vol} \otimes \omega^j$$

we require: $\rho^i{}_{j\mu} \in \mathbb{F}_2$.

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we require: $\rho^i{}_{j\mu} \in \mathbb{F}_2$.

For $\Omega^2 = D.\mathrm{Vol}$ we take 1-dimensional free module over D, with the basis denotes as Vol

Once we have specified at least Ω^2, we can:

- ask for our metric to be ‘quantum symmetric’ in the sense

$$\wedge(g) = 0$$

- Look for a *quantum Levi-Civita connection* (QLC):
  $\nabla g = T_\nabla = 0$

QLC connections and curvature on $\mathbb{F}_2\mathbb{Z}_3$

Recall: $g_D = \beta z^2\omega^1 \otimes \omega^1 + \beta z(\omega^1 \otimes \omega^2 + \omega^2 \otimes \omega^1) + \beta\omega^2 \otimes \omega^2$.

For $\beta = 1$ one of QLC's looks like this:

$$\nabla_{D.1.1}\omega^1 = z^2\omega^1 \otimes \omega^1 + (1+z)(\omega^1 \otimes \omega^2 + \omega^2 \otimes \omega^1) + \omega^2 \otimes \omega^2$$

$$\nabla_{D.1.1}\omega^2 = z^2\omega^1 \otimes \omega^1 + z\omega^1 \otimes \omega^2 + z^2\omega^2 \otimes \omega^1 + \omega^2 \otimes \omega^2$$

$$R_{\nabla_{D.1.1}}\omega^1 = \mathrm{Vol} \otimes \omega^1 + z^2\mathrm{Vol} \otimes \omega^2, \quad R_{\nabla_{D.1.1}}\omega^2 = z^2\mathrm{Vol} \otimes \omega^1;$$

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$$\nabla_{D.1.1}\omega^2 = z^2\omega^1 \otimes \omega^1 + z\omega^1 \otimes \omega^2 + z^2\omega^2 \otimes \omega^1 + \omega^2 \otimes \omega^2$$
$$R_{\nabla_{D.1.1}}\omega^1 = \mathrm{Vol} \otimes \omega^1 + z^2\mathrm{Vol} \otimes \omega^2, \quad R_{\nabla_{D.1.1}}\omega^2 = z^2\mathrm{Vol} \otimes \omega^1;$$

There are 3 more for this choice of β (none flat):

$$\nabla_{D.1.2}\omega^1 = z^2\omega^1 \otimes \omega^1 + z(\omega^1 \otimes \omega^2 + \omega^2 \otimes \omega^1) + \omega^2 \otimes \omega^2$$
$$\nabla_{D.1.2}\omega^2 = z^2\omega^2 \otimes \omega^1$$
$$R_{\nabla_{D.1.2}}\omega^1 = R_{\nabla_{D.1.2}}\omega^2 = \left(1+z^2\right)\mathrm{Vol} \otimes (\omega^1 + \omega^2);$$

$$\nabla_{D.1.3}\omega^1 = (z+z^2)\omega^1 \otimes \omega^1 + (1+z)\omega^1 \otimes \omega^2 + z\omega^2 \otimes \omega^1 + \left(1+z^2\right)\omega^2 \otimes \omega^2$$
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There are further 8 QLCs for $\beta = z$, $\beta = z^2$ (only 1 flat).

The Ricci tensor

$$\mathrm{Ricci} = ((\ ,\ ) \otimes \mathrm{id})(\mathrm{id} \otimes i \otimes \mathrm{id})R_\nabla$$

- 'lifting' bimodule map $i : \Omega^2 \to \Omega^1 \otimes_A \Omega^1$ such that $\wedge \circ i = \mathrm{id}$.
- When Ω^2 is 1-dim (with central basis Vol) then:

$$i(\mathrm{Vol}) = I_{ij}\omega^i \otimes \omega^j, \qquad I_{ij} \in A$$

for some central element of $\Omega^1 \otimes_A \Omega^1$ such that $\wedge(I) = \mathrm{Vol}$.

Then
$\mathrm{Ricci} = g_{ij}((\omega^i,\ ) \otimes \mathrm{id})(i \otimes \mathrm{id})R_\nabla\omega^j = g_{ij}(\omega^i, \rho^j{}_k I_{mn}\omega^m)\omega^n \otimes \omega^k.$

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Then

$\mathrm{Ricci} = g_{ij}((\omega^i,\ ) \otimes \mathrm{id})(i \otimes \mathrm{id}) R_\nabla \omega^j = g_{ij}(\omega^i, \rho^j{}_k I_{mn} \omega^m) \omega^n \otimes \omega^k$.

- I - not unique (we can add any functional multiple γg for $\gamma \in A$ if g is central and quantum symmetric)

For $D = \mathbb{F}_2\mathbb{Z}_3$ we take

$$i(Vol) = z^2\omega^2 \otimes \omega^1 + z\omega^2 \otimes \omega^2 + \gamma g$$

where $\gamma \in D, \quad \gamma = \gamma_1 + \gamma_2 z + \gamma_3 z^2.$

free parameters

Ricci tensor and scalar for $\mathbb{F}_2\mathbb{Z}_3$

Metric	QLC	Ricci (central for all γ_i)	$S = (\,,\,)(\text{Ricci})$	q. symmetric
$g_{D.1}$ $(\beta = 1)$	$\nabla_{D.1.2}$	Ricci $= 0$	$S = 0$	—
	$\left.\begin{matrix}\nabla_{D.1.1}\\ \nabla_{D.1.3}\\ \nabla_{D.1.4}\end{matrix}\right\}$	$\mathrm{Ricci} = \left(\gamma_3 + \gamma_2 z^2\right)\omega^1\otimes\omega^1 + \left(\gamma_2 z + \gamma_3 z^2\right)\omega^1\otimes\omega^2 + \left(\gamma_1 + z + \gamma_3 z^2\right)\omega^2\otimes\omega^1 + \left(1 + \gamma_3 z + \gamma_1 z^2\right)\omega^2\otimes\omega^2$	$\gamma_2 + \gamma_3 z$	$\gamma_1 = 0, \gamma_2 = 1:$ $\mathrm{Ricci} = (1+\gamma_3 z)z^2\omega^1\otimes\omega^1 + (1+\gamma_3 z)z\omega^1\otimes\omega^2 + (1+\gamma_3 z)z\omega^2\otimes\omega^1 + (1+\gamma_3 z)\omega^2\otimes\omega^2$
$g_{D.2}$ $(\beta = z)$	$\nabla_{D.2.4}$	Ricci $= 0$	$S = 0$	—
	$\left.\begin{matrix}\nabla_{D.2.1}\\ \nabla_{D.2.2}\\ \nabla_{D.2.3}\end{matrix}\right\}$	$\mathrm{Ricci} = \left(1 + \gamma_3 z + \gamma_1 z^2\right)\omega^1\otimes\omega^1 + \left(\gamma_3 + \gamma_1 z + z^2\right)\omega^1\otimes\omega^2 + \left(\gamma_1 z + (1+\gamma_2)z^2\right)\omega^2\otimes\omega^1 + \left(\gamma_1 + (1+\gamma_2)z\right)\omega^2\otimes\omega^2$	$1 + \gamma_2 + \gamma_1 z^2$	$\gamma_2 = 0 = \gamma_3:$ $\mathrm{Ricci} = (\gamma_1 + z)z^2\omega^1\otimes\omega^1 + (\gamma_1 + z)z\omega^1\otimes\omega^2 + (\gamma_1 + z)z\omega^2\otimes\omega^1 + (\gamma_1 + z)\omega^2\otimes\omega^2$
$g_{D.3}$ $(\beta = z^2)$	$\nabla_{D.3.1}$	Ricci $= 0$ (flat connection)	$S = 0$	—
	$\left.\begin{matrix}\nabla_{D.3.2}\\ \nabla_{D.3.3}\\ \nabla_{D.3.4}\end{matrix}\right\}$	$\mathrm{Ricci} = \left(\gamma_1 + (1+\gamma_2)z\right)\omega^1\otimes\omega^1 + \left(1 + \gamma_2 + \gamma_1 z^2\right)\omega^1\otimes\omega^2 + \left(\gamma_2 + \gamma_3 z\right)\omega^2\otimes\omega^1 + \left(\gamma_3 + \gamma_2 z^2\right)\omega^2\otimes\omega^2$	$1 + \gamma_3 z + \gamma_1 z^2$	never qsymm

For each metric one connection is Ricci flat for all lifts (indep. of γ_i).

$\underline{\dim}_{D.1} = \underline{\dim}_{D.2} = 1, \underline{\dim}_{D.3} = 0.$

The Einstein tensor

$$\begin{aligned} \mathrm{Eins} &= \mathrm{Ricci} + Sg \\ &= (\mathrm{Ricci}_{\mu ij} + S_\nu g_{\rho ij} V^{\nu\rho}{}_\mu) x^\mu \omega^i \otimes \omega^j \end{aligned}$$

with $\mathrm{Ricci}_{\mu ij},\ S_\nu,\ g_{\rho ij},\ V^{\nu\rho}{}_\mu \in \mathbb{F}_2$.

Note: the usual definition $\mathrm{Eins} = \mathrm{Ricci} - \frac{1}{2} Sg$ makes no sense over $\mathbb{F}_2$.
Here we have only two choices, $0, 1$, for the coefficient of Sg.

- We are interested in the values of $\mathrm{Eins} = \mathrm{Ricci} + Sg$
- If $\mathrm{Eins} \neq 0$ (as it would be classically for a 2D manifold) then we look for choices of γ when

$$\nabla \cdot \mathrm{Eins} = 0$$

- where $\nabla\cdot$ means to apply ∇ in the element of $\Omega^1 \otimes_D \Omega^1$ (same as for the metric) and then contract the first two factors with $(\ ,\ )$:

$$\nabla \cdot \mathrm{Eins} = \nabla \cdot \mathrm{Ricci} + ((\ ,\ ) \otimes \mathrm{id})(\mathrm{d}S \otimes g) = \nabla \cdot \mathrm{Ricci} + \mathrm{d}S.$$

The Einstein tensor on $\mathbb{F}_2\mathbb{Z}_3$

Metric	QLC	$\mathrm{Eins} = \mathrm{Ricci} + Sg$	Ricci qsymm	$\nabla\cdot\mathrm{Eins} = 0$
$g_{D.1}$	$\nabla_{D.1.2}$	$\mathrm{Eins} = 0$	—	—
	$\left.\begin{matrix}\nabla_{D.1.1}\\ \nabla_{D.1.3}\\ \nabla_{D.1.4}\end{matrix}\right\}$	$\mathrm{Eins} = (\gamma_1 + z(1+\gamma_2))\,\omega^2\otimes\omega^1 + \left(1+\gamma_2+\gamma_1 z^2\right)\omega^2\otimes\omega^2$	$\mathrm{Eins} = 0$	$\gamma_1 = 0$: $\mathrm{Eins} = (1+\gamma_2)\,z\omega^2\otimes\omega^1 + (1+\gamma_2)\,\omega^2\otimes\omega^2$
$g_{D.2}$	$\nabla_{D.2.4}$	$\mathrm{Eins} = 0$	—	—
	$\left.\begin{matrix}\nabla_{D.2.1}\\ \nabla_{D.2.2}\\ \nabla_{D.2.3}\end{matrix}\right\}$	$\mathrm{Eins} = (\gamma_2 + \gamma_3 z))\,\omega^1\otimes\omega^1 + \left(\gamma_3+\gamma_2 z^2\right)\omega^1\otimes\omega^2$	$\mathrm{Eins} = 0$	$\gamma_3 = 0$: $\mathrm{Eins} = \gamma_2\omega^1\otimes\omega^1 + \gamma_2 z^2\omega^1\otimes\omega^2$
$g_{D.3}$	$\nabla_{D.3.1}$	$\mathrm{Eins} = 0$ (flat connection)	—	—
	$\left.\begin{matrix}\nabla_{D.3.2}\\ \nabla_{D.3.3}\\ \nabla_{D.3.4}\end{matrix}\right\}$	$\mathrm{Eins} = \left(\gamma_2 z+\gamma_3 z^2\right)\omega^1\otimes\omega^1 + (\gamma_2+\gamma_3 z)\,\omega^1\otimes\omega^2 + \left(1+\gamma_2+\gamma_1 z^2\right)\omega^2\otimes\omega^1 + \left(\gamma_1 z+(1+\gamma_2)z^2\right)\omega^2\otimes\omega^2$	never qsymm	$\gamma_1 = 0 = \gamma_3$: $\mathrm{Eins} = \gamma_2 z\omega^1\otimes\omega^1 + \gamma_2\omega^1\otimes\omega^2 + (1+\gamma_2)\,\omega^2\otimes\omega^1 + (1+\gamma_2)z^2\omega^2\otimes\omega^2$

Metrics where $\underline{\dim} = 1$ have zero Einstein tensor when Ricci is lifted to be quantum symmetric.
The metric $g_{D.3}$ where $\underline{\dim} = 0$ has two lifts for the non-flat connections with $\nabla\cdot\mathrm{Eins} = 0$ and $S = 1$.

Digital Quantum Geometries on $D = \mathbb{F}_2\mathbb{Z}_3$:

- for each metric one connection is Ricci flat for all lifts (and only actually flat for $g_{D.3}$)
- and the other three connections all have the same Ricci curvature
- when Ricci is quantum symmetric (choice of γ_i) then $\mathrm{Eins} = 0$
- we can chose the lift so that $\nabla \cdot \mathrm{Eins} = 0$

$$g_{D.1}: \quad \gamma_1 = \gamma_3 = 0, \gamma_2 = 1, \quad \mathrm{Ricci} = g_{D.1}, \quad S = 1, \quad \nabla\cdot\mathrm{Ricci} = 0, \quad \mathrm{Eins} = 0$$

$$g_{D.2}: \quad \gamma_1 = \gamma_2 = \gamma_3 = 0, \quad \mathrm{Ricci} = g_{D.2}, \quad S = 1, \quad \nabla\cdot\mathrm{Ricci} = 0, \quad \mathrm{Eins} = 0$$

$$g_{D.3}: \quad \gamma_1 = \gamma_3 = 0, \quad S = 1, \quad \nabla \cdot \mathrm{Ricci} = \nabla \cdot \mathrm{Eins} = 0, \quad \mathrm{Eins} \neq 0$$

- the last case is unusual in that classically the Einstein tensor in 2D would vanish , but this is also the 'unphysical' case where $\underline{\mathrm{dim}_{\mathrm{D.3}}} = 0$.

- Similar results were obtained for two other (commutative) algebras $B = \mathbb{F}_2(\mathbb{Z}_3)$ and $F = \mathbb{F}_8$.

- We have also investigated the properties of the geometric Laplacians:

$$\Delta = (\ ,\ )\nabla \mathrm{d} : \qquad A \to A$$

- For algebras A, C, E, G there are no invertible central metrics for the universal calculus.

- All results - see S.Majid, A.P., J.Phys. A 2019 (in press) [arXiv:1807.08492].

Summary

- We have mapped out the **landscape** of all reasonable up to **2D quantum geometries** over the field $\mathbb{F}_2$ on unital algebras of dimension $n \leq 3$.
- In $n = 3$ with 2-dim Ω^1 we find that only **three** of the six algebras, namely B$= \mathbb{F}_2(\mathbb{Z}_3)$, **D**$= \mathbb{F}_2\mathbb{Z}_3$, F$= \mathbb{F}_8$, meet our full requirements on the calculus including Ω^2 as top form degree 2 and existence of a quantum symmetric metric.
- The interesting ones up to this dimension have **commutative coordinate algebras**

Conclusions

- For each of them we find an essentially **unique calculus and a unique quantum metric** up to an invertible functional factor
- When the quantum metrics admit QLC connections, each pair produces **'digital quantum Riemannian geometry'** of which most are not flat in the sense of non-zero Riemann curvature R_∇
- For the Ricci tensor: we have found 2, 2, 5 (for alg. B, D, F resp.) - a **total of 9 interesting Ricci flat but not flat quantum geometries** over $\mathbb{F}_2$.
- These deserve more study in view of the important role of Ricci flat metrics in classical GR (as vacuum solutions of Einstein's equations).

Perspectives

- Finite field setting allows one to test definitions and conjectures - full classification possible.
- Quantum gravity is normally seen as a weighted 'sum' over all possible metrics
- once we have a good handle on the moduli of classes of small $\mathbb{F}_{p^d}$ quantum Riemannian geometries, we could consider quantum gravity, for example as a weighted sum over the moduli space of them much as in lattice approximations, but now finite.

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Thank you for your attention!