LABORATOIRE D'ANNECY-LE-VIEUX DE PHYSIQUE THÉORIQUE

Some aspects of Non Equilibrium Quantum Field Theory

Paul Sorba

(LAPTh- CNRS- France)

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M. Mintchev (Pisa) and ***L. Santoni*** (Columbia Univ.- New York)

paul.sorba@lapth.cnrs.fr

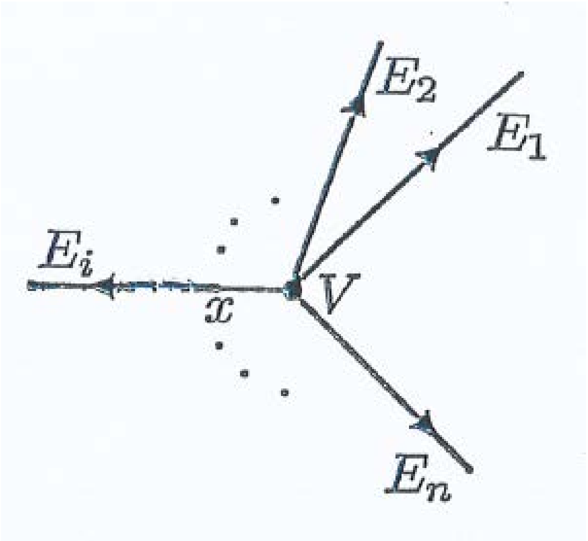
Some developments of Non Equilibrium Quantum Field Theory considering Quantum wires in the form of Star Graphs

Continuation of a program started about 20 years ago with M.Mintchev and other collab. (E.Ragoucy, M.Burrello, B.Bellazzini, L.Santoni):

i) Algebraic framework for dealing with defects in 1+1dim., introducing the ''reflection-transmission'' or ''R-T'' algebras, powerful approach to integrable systems with impurities.

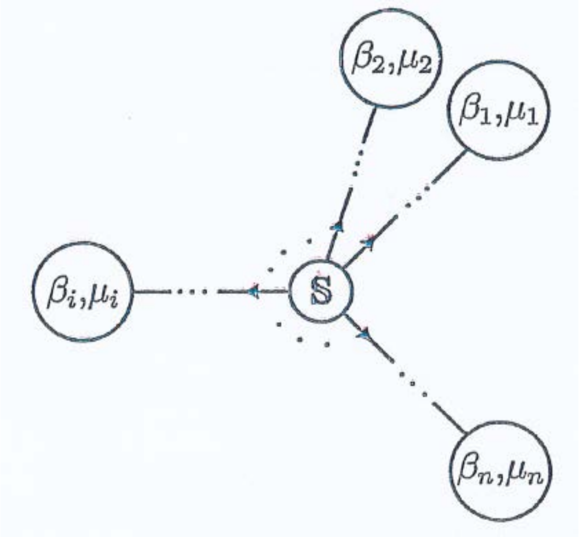
ii) With spectral theory of Schrodinger operator on quantum graphs, formalism for explicit computations, complete classification of boundary conditions and determination of physical quantities i.e. conductance in different models.

(cf. ''Quantum Wires'' seminar at MPHYS 6 (2010))



Quantum networks first applied to electron transport in organic molecules, then appeared in interacting 1 dim. electron gaz. Applications due to rapid progress in **nanoscale quantum devises**.

iii) Non Equilibrium Quantum Systems with thermal reservoirs at the edges of the network.



What we have done:

- an explicit construction in field theory of Non Equilibrium Steady States - or NESS,

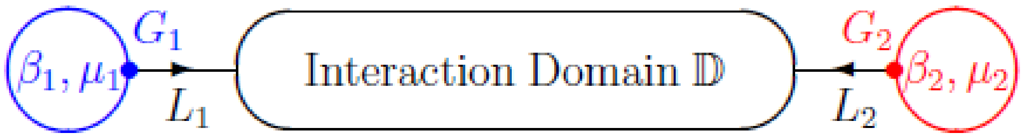
- a study of microscopic features of quantum transport and entropy production.

Based on:

M. Mintchev, L. Santoni, P.S. [1] J.Phys.A: Math.Theor.48 (2015) 285002;

[2] Phys.Rev.E96, 052124 (20170; [3] Annalen der Physyk 530, 201800170 (2018)

Quantum transport in systems of the type:



- 2 oriented leads L_i conecting via the gates G_i 2 heat reservoirs

$$R_1 = \{\beta_1, \mu_1\} \text{ (cold) and } R_2 = \{\beta_2, \mu_2\} \text{ (hot)}$$

with the interaction domain $\mathbb{D}$;

- R_i have a large enough capacity such that the particle emission/absorption through G_i does not change the parameters $\{\beta_i, \mu_i\}$;

Realistic (essentially 1+1 dimensional) systems modelled by this setup:

- fermionic junctions - quantum nanowires;
- bosonic junctions - ultacold Bose gasses in one-dimensional laser traps;
- anyonic junctions - quantum Hall edges;

Previous results: mean value of charge and heat currents and the associated noise;

Goal of this study: develop a systematic microscopic approach to explore quantum fluctuations of both currents and entropy production.

Basic microscopic aspects:

- Nontrivial particle ($\mu_1 \neq \mu_2$) and heat ($\beta_1 \neq \beta_2$) currents are flowing in the system
- The microscopic origin of these currents are three kinds of processes:
  - emission and absorption of n particles from the same reservoir $\Longrightarrow$ vanishing entropy production
  - emission of n particles from R_2 and absorption in R_1 $\Longrightarrow$ positive entropy production;
  - emission of n particles from R_1 and absorption in R_2 $\Longrightarrow$ negative entropy production;
- Each of these processes is characterised by some probability p_n:

$$P = \{p_n \,:\, n = 0, \pm 1, \pm 2, \ldots\},$$

- Universal feature of P: fully codifies
  - all kinds of quantum transport: charge, energy and heat;
  - the entropy production - key quantity quantifying the departure from equilibrium;
  - all the quantum fluctuations.

Problem: derive the set P - QFT approach.

QFT strategy - three steps:

- derive the n-point correlation functions

$$w_n[\zeta](t_1, x_1, \dots, t_n, x_n) = \langle \zeta(t_1, x_1, i) \cdots \zeta(t_n, x_n) \rangle_\Omega \,, \qquad n = 1, 2, \dots$$

of the particle current and the entropy production operators:

$$\zeta(t_1, x_1, i) = j(t_1, x_1, i)\,, \qquad \zeta(t_1, x_1, i) = \dot{S}(t_1, x_1)\,;$$

$w_n[\zeta]$ are the moments $\mathcal{M}_n[\zeta]$ of a probability distribution $\{\mathcal{D},\, \varrho[\zeta]\}$

$$w_n[\zeta] = \int_{\mathcal{D}} \mathrm{d}\sigma\, \sigma^n \varrho[\zeta](\sigma) \equiv \mathcal{M}_n[\zeta]\,;$$

- reconstruct $\varrho[\zeta]$ from $\mathcal{M}_n[\zeta]$ (solve the moment problem);
- extract from the distributions $\varrho[\zeta]$ the microscopic information:

$$\{p_n \;:\; n = 0, \pm 1, \pm 2, \dots\}\,, \quad R_1 \leftrightarrow R_2 \text{ emission} - \text{absorption probabilities}$$
$$\{\sigma_n \;:\; n = 0, \pm 1, \pm 2, \dots\}\,, \quad \text{associated entropy production}$$

Plan of the talk - above points + applications:

Zoom in on the first step:

The derivation of the correlation functions

$$w_n[j_i](t_1, x_1, ..., t_n, x_n) = \langle j(t_1, x_1, i) \cdots j(t_n, x_n, i)\rangle_\Omega \,,$$

$$w_n[\dot{S}](t_1, x_1, ..., t_n, x_n) = \langle \dot{S}(t_1, x_1) \cdots \dot{S}(t_n, x_n)\rangle_\Omega \,,$$

is based on two ingredients:

- the observables in terms of the basic fields of the theory (e.g. fermions, bosons, anyons,...):
  - $j(t, x, i)$ - particle current (local information about the lead L_i);
  - $\dot{S}(t, x)$ - entropy production (global information about the whole system);

  $j, \dot{S}, ... \in \mathcal{A}$ - the algebra all observables of the system
- a state Ω for computing the expectation values $\langle \cdots \rangle_\Omega$
  - in other words, one should fix a representation $\pi : \mathcal{A} \to \mathcal{H}$;
  - choose π which fits better the physical situation in consideration;
  - our case - generalise the Gibbs representation to the case with two heat baths R_i interacting with $\mathbb{S}$

  Landauer-Bütticker representation: $\pi_{LB}[\beta_i, \mu_i, \mathbb{S}]$;
- symmetries play fundamental role in this step.

Symmetry content - continuous symmetries:

We shall consider systems preserving:

- the particle number N: $\Longrightarrow$ $\partial_t j_t - \partial_x j_x = 0$
- the total energy E: $\Longrightarrow$ $\partial_t \theta_{tt} - \partial_x \theta_{xt} = 0$
- $\dot{N} = 0$: $\Longrightarrow$ $\sum_{i=1}^{2} j_x(t, G_i, i) = 0$
- $\dot{E} = 0$: $\Longrightarrow$ $\sum_{i=1}^{2} \theta_{xt}(t, G_i, i) = 0$

With these two very general assumptions the junction operates as dissipationless converter of heat to chemical energy or vice versa.

- the reason is that the heat current Tapez une équation ici. $q_x = \theta_{xt} - \mu_i j_x$
and the chemical potential current $k_x = \mu_i j_x$
are locally conserved but

$$\sum_{i=1}^{2} q_x(t, G_i, i) = -\sum_{i=1}^{2} k_x(t, G_i, i) = (\mu_2 - \mu_1) j_x(t, G_1, 1) \neq 0 \,, \quad \mu_1 \neq \mu_2 \,;$$

- energy conversion - controlled by the operator: $\dot{Q} = -\sum_{i=1}^{2} q_x(t, G_i, i)$;
- let $\Psi \in \mathcal{H}$ be any state of the system;

$$\langle \dot{Q} \rangle_\Psi < 0 \,, \qquad \text{heat energy} \longrightarrow \text{chemical energy} \,,$$
$$\langle \dot{Q} \rangle_\Psi > 0 \,, \qquad \text{chemical energy} \longrightarrow \text{heat energy} \,.$$

- energy conversion: universal - holds for any dynamics in $\mathbb{D}$ respecting the symmetries;

Symmetry content - discrete symmetries:

- time reversal - essential for entropy production;
- time reversal operation - anti-unitary operator T, such that

$$T\, j(t,x,i)\, T^{-1} = -j(-t,x,i)\,;$$

- suppose that: $\langle j(t,x,i)\rangle_\Omega \neq 0\,;$
- time translation invariance $\Longrightarrow \langle j(t,x,i)\rangle_\Omega$ is t-independent;

$$\Downarrow$$

$$T\Omega \neq \Omega$$

- origin of non-vanishing entropy production

$$\langle \dot{S}(t,x,i)\rangle_\Omega \neq 0\,, \qquad \dot{S} = -\sum_{i=1}^{2} \beta_i q(t,x,i)\,;$$

- QFT effect: no need of explicit T-breaking via dissipation:
  - external classical fields (magnetic field);
  - other quantum systems.

Summarising, systems of the type

which preserve

- total energy
- particle number

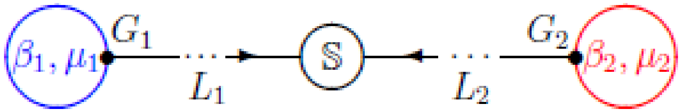
have the following universal model independent features:

- convert heat to chemical energy or vice versa without dissipation;
- produce entropy without explicit time reversal breaking.

Rest of the talk: illustrate these general features on concrete examples controlling:

- the spectrum of H - to be sure that there is no dissipation;
- time reversal invariance $THT^{-1} = H$ - to be sure that the entropy production is a consequence of spontaneous breaking.

Example - fermionic/bosonic Schrödinger junction:



- $\mathbb{D}$ shrinks to a point $x = 0$ - exactly solvable model:
  - mathematically very clean;
  - illustrates the universal aspects,
  - illustrates the impact of statistics;
- dynamics on the leads:

$$\left(\mathrm{i}\partial_t + \frac{1}{2m}\partial_x^2\right)\psi(t,x,i) = 0\,, \qquad (x < 0\,:\, i = 1,2) \text{ local coordinates on } L_i\,;$$

- statistics - fermionic $+$ or bosonic $-$ junction

$$[\psi(t,x,i)\,,\, \psi^*(t,y,j)]_\pm = \delta_{ij}\,\delta(x-y)\ :$$

- point-like interaction - boundary condition ($\mathbb{U} \in U(2)$, λ-free parameter):

$$\lim_{x\to 0^-} \sum_{j=1}^{2} \left[\lambda(\mathbb{I}-\mathbb{U})_{ij} + \mathrm{i}(\mathbb{I}+\mathbb{U})_{ij}\partial_x\right]\psi(t,x,j) = 0\,;$$

- this is the most general b.c. ensuring the self-adjointness of the Hamiltonian;
- interaction - the associated scattering matrix is (Kostrykin-Schrader 2000):

$$\mathbb{S}(k) = -\frac{[\lambda(\mathbb{I}-\mathbb{U}) - k(\mathbb{I}+\mathbb{U})]}{[\lambda(\mathbb{I}-\mathbb{U}) + k(\mathbb{I}+\mathbb{U})]}\,;$$

The solution in absence of bound states of S:

$$\psi(t,x,i) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-i\omega(k)t - ikx} a_i(k), \qquad \omega(k) = \frac{k^2}{2m}$$

- $\{a_i(k),\ a_j^*(p)\}$ generate the (anti)commutation relation algebras $\mathcal{A}_\pm$ and satisfy

$$[a_i(k), a_j^*(p)]_\pm = 2\pi\left[\delta_{ij}\delta(k-p) + S_{ij}(k)\delta(p+k)\right].$$

$$\textit{constraints:}\quad a_i(k) = S_{ij}(k)\, a_i(-k)\ ;\ a_i^*(k) = a_j^*(-k)\, S_{ji}(-k)$$

- interaction codified in the algebra - greatly simplifies the analysis;
- observables:
  - particle current
    $j_x(t,x,i) = \frac{i}{2m}[\psi^*(\partial_x\psi) - (\partial_x\psi^*)\psi](t,x,i)$;
  - energy current
    $\theta_{xt}(t,x,i) =$
    $\frac{1}{4m}[(\partial_t\psi^*)(\partial_x\psi) + (\partial_x\psi^*)(\partial_t\psi) - (\partial_t\partial_x\psi^*)\psi - \psi^*(\partial_t\partial_x\psi)](t,x,i)$;
  - heat current $q_x(t,x,i) = \theta_{xt}(t,x,i) - \mu_i j_x(t,x,i)$;
  - entropy production operator $S(t,x) = -\sum_{i=1,2}\beta_i q_x(t,x,i)$
- fix a representation of $\mathcal{A}_\pm$.

Algebraic construction of the NESS:

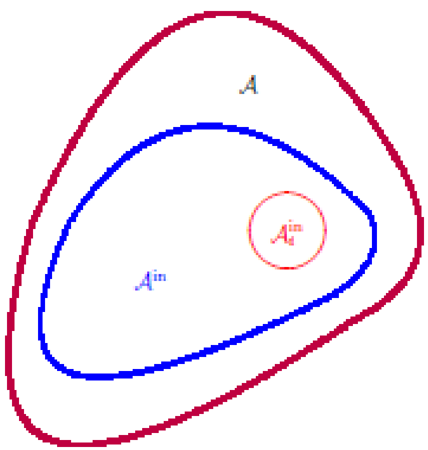
Consider the incoming sub-algebra $\mathcal{A}^{\rm in}_{\pm,i} = \{a_i(k),\, a_i^*(k) \,:\, k > 0\}$ associated to R_i and perform the following three steps:

- take the Gibbs state Ω_{β_i,μ_i} over $\mathcal{A}^{\rm in}_{\pm,i}$;

$$\left(\Omega_{\beta_i,\mu_i}\,,\, \mathcal{O}\,\Omega_{\beta_i,\mu_i}\right) \equiv \langle \mathcal{O} \rangle_{\beta_i,\mu_i} = \frac{1}{Z} \mathrm{Tr}\left[\mathrm{e}^{-K_i} \mathcal{O}\right]\,, \qquad Z = \mathrm{Tr}\left[\mathrm{e}^{-K_i}\right]\,,$$

$$K_i = \beta_i(h_i - \mu_i q_i)\,, \quad h_i = \int_0^\infty \frac{\mathrm{d}k}{2\pi} \omega(k) a_i^*(k) a_i(k)\,, \quad q_i = \int_0^\infty \frac{\mathrm{d}k}{2\pi} a_i^*(k) a_i(k)\,.$$

- perform the tensor product $\Omega^{\rm in}_{\beta,\mu} = \bigotimes_{i=1}^n \Omega_{\beta_i,\mu_i}$;
- extend $\Omega^{\rm in}_{\beta,\mu}$ by linearity to a state $\Omega_{\beta,\mu}$ on the whole algebra $\mathcal{A}_\pm$ using the scattering relations



$$a_i(k) = \sum_{j=1}^n \mathbb{S}_{ij}(k) a_j(-k)\,, \qquad a_i^*(k) = \sum_{j=1}^n a_j^*(-k) \mathbb{S}_{ji}^*(k)\,;$$

The LB non-equilibrium steady state:

- the Landauer (1970)-Büttiker (1986) (LB) representation π_{LB} of $\mathcal{A}_\pm$;
- the basic correlator in the Fermi/Bose ($\pm$) LB state $\Omega_{LB}^{\pm}$ is:

$$\langle a_{l_1}^*(k_1)a_{m_1}(p_1)\cdots a_{l_n}^*(k_n)a_{m_n}(p_n)\rangle_{\mathrm{LB}}^{\pm}\,, \qquad k_i>0\,, p_i>0$$

- introduce the matrix

$$\mathbb{M}_{ij}^{\pm}=\begin{cases}2\pi\delta(k_i-p_j)\delta_{l_i m_j}d_{l_i}^{\pm}[\omega(k_i)]\,, & i\leq j\,,\\ \mp 2\pi\delta(k_i-p_j)\delta_{l_i m_j}\left(1\mp d_{l_i}^{\pm}[\omega(k_i)]\right)\,, & i>j\,,\end{cases}$$

$$d_l^{\pm}(\omega)=\frac{1}{\mathrm{e}^{\beta_l(\omega-\mu_l)}\pm 1}\,, \qquad (\text{for bosons } \mu_l<0)$$

  being the Fermi/Bose distribution of the reservoir R_l.
- then,

$$\langle a_{l_1}^*(k_1)a_{m_1}(p_1)\cdots a_{l_n}^*(k_n)a_{m_n}(p_n)\rangle_{\mathrm{LB}}^{\pm}=\begin{cases}\det\,[\mathbb{M}^+]\,,\\ \mathrm{perm}\,[\mathbb{M}^-]\,,\end{cases}$$

- permanent:

$$\mathrm{perm}\,[\mathbb{M}]=\sum_{\sigma_i\in\mathcal{P}_n}\prod_{i=1}^{n}\mathbb{M}_{i\sigma_i}\,, \qquad \mathcal{P}_n - \text{set of all permutations of } n \text{ elements}\,.$$

One-point correlators in the LB state:

- $\langle j(t,x,1)\rangle^{\pm}_{\rm LB} = -\langle j(t,x,2)\rangle^{\pm}_{\rm LB} = \int_0^\infty \frac{{\rm d}\omega}{2\pi} |\mathbb{S}_{12}(\sqrt{2m\omega})|^2 [d_1^{\pm}(\omega) - d_2^{\pm}(\omega)] \neq 0\, ;$
  - t-independent because of energy conservation;
  - x-independent because of particle number conservation;
  - $\neq 0$ provided that $\beta_1 \neq \beta_2$ and/or $\mu_1 \neq \mu_2$ (away from equilibrium);
- time reversal is spontaneously broken;
  - the origin of nontrivial entropy production in the system;

$$\langle \dot{S}(t,x)\rangle^{\pm}_{\rm LB} = \int_0^\infty \frac{{\rm d}\omega}{2\pi} |\mathbb{S}_{12}(\sqrt{2m\omega})|^2 [\gamma_2(\omega) - \gamma_1(\omega)][d_1^{\pm}(\omega) - d_2^{\pm}(\omega)] \geq 0\, ;$$

$$\gamma_i \equiv \beta_i(\omega - \mu_i)$$

  - the mean entropy production is non-negative;
  - on this ground a junction converting heat in chemical energy can be considered as a "heat engine" (mechanical energy $\to$ chemical energy);
  - this microscopic "engine" consists of a single point-like quantum defect;
  - the question of efficiency - discussion in what follows.

Correlation functions with $n \geq 2$ - quantum fluctuations:

- properties of $w^{\pm}_{n\geq 2}[\dot{S}]$ - depend on x_i and the time differences $\hat{t}_i \equiv t_i - t_{i+1}$:
- zero-frequency limit

$$\lim_{\nu \to 0^+} \int_{-\infty}^{\infty} \mathrm{d}\hat{t}_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}\hat{t}_{n-1} \mathrm{e}^{-\mathrm{i}\nu(\hat{t}_1 + \cdots \hat{t}_{n-1})} w^{\pm}_n[\dot{S}](t_1, x_1, \dots, t_n, x_n)$$
$$= \int_0^{\infty} \frac{\mathrm{d}\omega}{2\pi} \mathcal{M}^{\pm}_n[\dot{S}](\omega) \qquad ,$$

- significant simplification - x_i-independendence;
- comments:
  - at low frequencies the fluctuation are integrated over long period of time;
  - regime explored in full counting statistics and transport experiments (weak signals);
  - the bound states of $\mathbb{S}$ do not contribute in the limit $\nu \to 0$;

$\mathcal{M}^{\pm}_n[\dot{S}]$ are the moments of the probability distribution $\varrho[\dot{S}]$ we are looking for.

The moments $\mathcal{M}_n^{\pm}[\dot{S}]$:

- the explicit form of $\langle a^*_{l_1}(k_1) a_{m_1}(p_1) \cdots a^*_{l_n}(k_n) a_{m_n}(p_n)\rangle^{\pm}_{\mathrm{LB}}$ implies

$$\mathcal{M}_n^{\pm}[\dot{S}] = \begin{cases} \gamma_{21}^n(\omega)\,\mathbf{det}[\mathbb{D}^+(\omega; l_1, \ldots, l_n)]\,, \\ \gamma_{21}^n(\omega)\,\mathbf{perm}[\mathbb{D}^-(\omega; l_1, \ldots, l_n)]\,, \end{cases}$$

with $\gamma_{ij}(\omega) = \gamma_i - \gamma_j = (\beta_i - \beta_j)\omega - (\beta_i \mu_i - \beta_j \mu_j)$

- $\gamma_{ij}(\omega)$ - basic dimensionless parameter characterising the transport of a particle from the reservoir R_i to R_j;

$$\mathbb{D}_{ij}^{\pm}(\omega; l_1, \ldots, l_n) = \begin{cases} \mathbb{J}_{l_j l_i}(\omega) d_{l_j}^{\pm}(\omega)\,, & i \leq j\,, \\ \mp \mathbb{J}_{l_j l_i}(\omega)\left[1 \mp d_{l_j}^{\pm}(\omega)\right]\,, & i > j\,, \end{cases}$$

with
$$\mathbb{J}_{11}(\omega) = -\mathbb{J}_{22}(\omega) = |\mathbb{S}_{12}(\sqrt{2m\omega})|^2 \equiv \tau(\omega)\,,$$
$$\mathbb{J}_{12}(\omega) = \overline{\mathbb{J}}_{21}(\omega) = -\mathbb{S}_{11}(\sqrt{2m\omega})\,\overline{\mathbb{S}}_{12}(\sqrt{2m\omega})\,,$$

- $\tau(\omega)$ - transmission probability ($\tau(\omega) = 0$ - isolated leads);

Main observation: using the above representation, one can prove ([1], [2] for details) the bound

$$\mathcal{M}_n^{\pm}[\dot{S}] \geq 0\,, \qquad n =, 1, 2, \ldots$$

Comments about the entropy production bound:

$$\mathcal{M}_n^{\pm}[\dot{S}] \geq 0$$

- the quantum fluctuations do not alter the behaviour of the mean value $\mathcal{M}_1^{\pm}[\dot{S}]$;
- the bound holds for both fermions and bosons;
- the equality $=$ holds only at equilibrium;
- the entropy production bound is not shared by the particle and heat current fluctuations;
- tempting interpretation - a quantum version of the second law $\dot{S}_{\rm cl} \geq 0$ (further comments on that later);
- in this spirit the bound can be used for selecting non-equilibrium states;
- uncover the microscopic origin of the bound - investigate the probability distribution $\varrho^{\pm}[\dot{S}]$ generating the moments $\mathcal{M}_n^{\pm}[\dot{S}]$;
- start with the moment generating function $\chi^{\pm}[\dot{S}]$.

The moment generating function $\chi^{\pm}[\dot{S}]$:

- the whole information from the moments is stored by the moment generating function

$\chi^{\pm}[\dot{S}](\lambda) = \sum_{n=0}^{\infty} \frac{(\mathrm{i}\lambda)^n}{n!} \mathcal{M}_n^{\pm}[\dot{S}]$; from now on the ω-dependence is implicit

- using the determinant/permanent representation of $\mathcal{M}_n^{\pm}[\dot{S}]$ one finds:

$$\chi^{+}[\dot{S}](\lambda) = 1 + \mathrm{i}c_1^{+}\sqrt{\tau}\sin(\lambda\gamma_{21}\sqrt{\tau}) + c_2^{+}\left[\cos(\lambda\gamma_{21}\sqrt{\tau}) - 1\right] ,$$

$$\chi^{-}[\dot{S}](\lambda) = \frac{1}{1 - \mathrm{i}c_1^{-}\sqrt{\tau}\sin(\lambda\gamma_{21}\sqrt{\tau}) - c_2^{-}\left[\cos(\lambda\gamma_{21}\sqrt{\tau}) - 1\right]} ,$$

- dependence on:
  - the heat baths distributions via:

$$c_1^{\pm} \equiv d_1^{\pm} - d_2^{\pm} , \quad c_2^{\pm} \equiv d_1^{\pm} + d_2^{\pm} \mp 2d_1^{\pm}d_2^{\pm} ;$$

  - the entropy production unit $\gamma_{12}\sqrt{\tau} = [(\beta_2 - \beta_1)\omega - (\beta_2\mu_2 - \beta_1\mu_1)]\sqrt{\tau}$;
  - transmission probability τ characterising the interaction.

The probability distribution $\varrho^{\pm}[\dot{S}]$:

- the final step towards the probability distribution $\varrho^{\pm}[\dot{S}]$ is the Fourier transform

$$\varrho^{\pm}[\dot{S}](\sigma) = \int_0^{\infty} \frac{\mathrm{d}\lambda}{2\pi} \mathrm{e}^{-\mathrm{i}\lambda\sigma} \chi^{\pm}[\dot{S}](\lambda)\,,$$

- $\chi^{\pm}[j_i]$ is a periodic function in λ with period $2\pi/\sqrt{\tau}$;
- accordingly, the Fourier transform is a superposition of δ-functions - Dirac comb:
  - the fermionic comb has three "teeth"

$$\varrho^{+}[\dot{S}](\sigma) = \sum_{k=-1}^{1} p_k^{+}\,\delta(\sigma - k\gamma_{21}\sqrt{\tau})\,.$$

    because only single particle processes are allowed by Pauli's principle since the energy ω is fixed and there is no degeneracy in spin and momentum;
  - the bosonic comb has infinite "teeth" because multiparticle processes are allowed;

$$\varrho^{-}[\dot{S}](\sigma) = \sum_{k=-\infty}^{\infty} p_k^{-}\,\delta(\sigma - k\gamma_{21}\sqrt{\tau})\,,$$

  - $p_k^{\pm}$ are the probabilities we are looking for;
  - $k\gamma_{21}\sqrt{\tau}$ is the entropy production associated with $p_k^{\pm}$.

Explicit form of the sets $P^{\pm}$:

- fermionic probabilities: $p^{+}_{\pm 1} = \frac{1}{2}\left(c_2^{+} \mp c_1^{+}\sqrt{\tau}\right)\,, \quad p^{+}_{0} = 1 - c_2^{+}$

  with: $c_1^{+} \equiv d_1^{+} - d_2^{+}\,, \quad c_2^{+} \equiv d_1^{+} + d_2^{+} - 2d_1^{+}d_2^{+};$

  $$p_k^{+} \geq 0\,, \qquad \sum_{k=-1}^{1} p_k^{+} = 1\,;$$

- bosonic probabilities: $p^{-}_{\pm n} = \frac{b_{\pm}^{n}}{1+c_2^{-}}\, {}_2F_1\left[\frac{1+n}{2}, \frac{2+n}{2}, n+1, 4b_{+}b_{-}\right]$

  with $b_{\pm} = \frac{(c_2^{-} \pm c_1^{-}\sqrt{\tau})}{2(1+c_2^{-})}\,, \quad c_1^{-} \equiv d_1^{-} - d_2^{-}\,, \quad c_2^{-} \equiv d_1^{-} + d_2^{-} + 2d_1^{-}d_2^{-}$

  $$p_k^{-} \geq 0\,, \qquad \sum_{k=-\infty}^{\infty} p_k^{-} = 1\,;$$

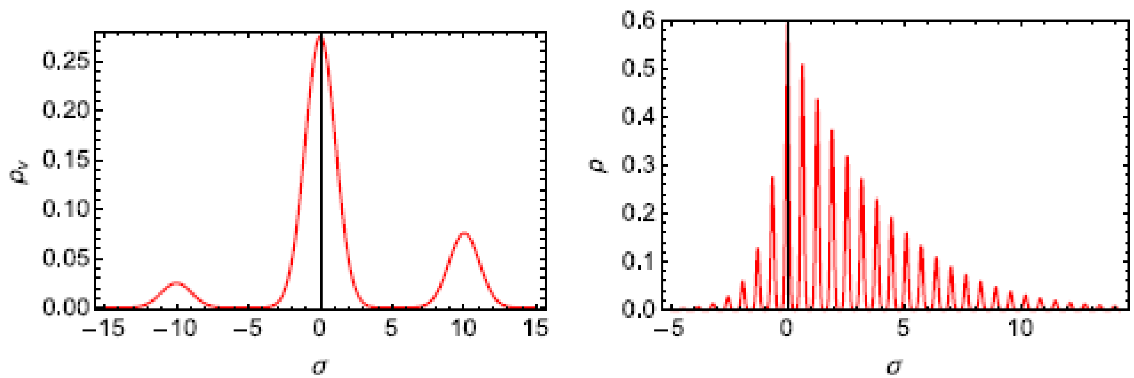
- $p_k^{\pm}$ - probabilities of the basic microscopic processes of emission and absorption in terms of the Dirac/Bose distributions $d_i^{\pm}$ and the interaction τ;
- $p_k^{\pm}$ carry the information of all quantum fluctuations;
- relative simplicity of p_k^{+} with respect to p_k^{-} - consequence of Pauli principle.

The smeared distributions $\varrho^{\pm}[\dot{S}]$:

- since $\varrho^{\pm}[\dot{S}]$ are singular, it is convenient for physical considerations to perform a smearing;

$$\delta(\sigma) \longmapsto \delta_{\alpha}(\sigma) = \frac{\alpha}{\sqrt{\pi}} \mathrm{e}^{-\alpha^2\sigma^2} \text{ with } \alpha > 0;$$

- one gets the smeared distributions $\varrho^{\pm}_{\alpha}[\dot{S}]$ approaching $\varrho^{\pm}[\dot{S}]$ for $\alpha \to \infty$.



- the peaks are associated with the fundamental emission-absorption processes;
- the predominant process is the emission and reabsorption by the same reservoir;
- the right and the left peaks are symmetric with respect to $\sigma = 0$ and the right ones dominate $\Longrightarrow \mathcal{M}^{\pm}_{2k-1}[\dot{S}] \geq 0$.
- $\mathcal{M}^{\pm}_{2k}[\dot{S}] \geq 0$ because $\varrho^{\pm}[\dot{S}]$ is a true distribution and not a quasi-probability (in the sense of Wigner) distribution.

Application - fluctuation relation:

- conventional fluctuation relation (Evans, Searles, Gallavotti, Cohen, Crooks, Jarzynski,...)

$$\frac{P[-\dot{S}]}{P[\dot{S}>0]} \sim e^{-\dot{S}} \quad \underset{\dot{S}\to\infty}{\longrightarrow} \quad 0$$

- the fluctuation relation in our system;

$$\gamma_i = \beta_i(\omega-\mu_i): \qquad 0<\dot{S}\longrightarrow\infty \Longleftrightarrow \text{keep } \gamma_1 \text{ fixed and take } \gamma_2 \longrightarrow \infty\,;$$

- fermions: $\frac{P_f[-\dot{S}]}{P_f[\dot{S}>0]} = \frac{p^+_{-1}(\gamma_1,\gamma_2,\tau)}{p^+_1(\gamma_1,\gamma_2,\tau)} \underset{\dot{S}\to\infty}{\longrightarrow} \frac{1-\sqrt{\tau}}{1+\sqrt{\tau}} \underset{\tau\to 1}{\longrightarrow} 0$
- bosons:

$$\frac{P_b[-\dot{S}]}{P_b[\dot{S}>0]} = \frac{\sum_{k=1}^{\infty} p^+_{-k}(\gamma_1,\gamma_2,\tau)}{\sum_{k=1}^{\infty} p^+_k(\gamma_1,\gamma_2,\tau)} \underset{\dot{S}\to\infty}{\longrightarrow} \frac{(\tau+e^{\gamma_1}-1)-\sqrt{\tau(\tau+e^{\gamma_1}-1)}}{1+\sqrt{\tau}} \underset{\tau\to 1}{\longrightarrow} 0$$

- conventional fluctuation relation - recovered in the homogeneous limit $\tau\to 1$;
- impact of defects - needs further investigation.

Application - efficiency of quantum transport:

- mean value efficiency (Casati, Benenti, Saito, Prosen, Seifert,...)
- use that the mean entropy production is non-negative

$$\langle \dot{S} \rangle \geq 0;$$

- on this ground a junction converting heat in chemical energy, namely

$$\langle \dot{Q} \rangle < 0$$

  can be considered as a heat engine (mechanical energy $\rightarrow$ chemical energy);
- let $\mathcal{K}_+$ be the set of positive heat currents. Then

$$\eta = \frac{-\langle \dot{Q} \rangle}{\sum_{i \in \mathcal{K}_+} \langle q(i) \rangle},$$

- $\langle \dot{S} \rangle \geq 0$ implies the Carnot bound:

$$0 < \eta < 1 - r \equiv \eta_C, \qquad r \equiv \frac{\beta_1}{\beta_2}, \qquad \beta_2 \geq \beta_1;$$

- limitation of η: does not take into account the essence of quantum physics - the fluctuations.

Efficiency beyond the mean value description:

- concept of efficiency which takes into account all quantum fluctuations;
- take advantage of the microscopic picture which provides separately the total rate of positive and negative entropy productions ($\gamma_{21} > 0$):

$$\sigma[\dot{S} > 0] = \int_0^\infty \mathrm{d}\omega\, \gamma_{21} \sqrt{\tau} \sum_{k=1}^{\infty} k\, p_k\,, \qquad \sigma[\dot{S} < 0] = -\int_0^\infty \mathrm{d}\omega\, \gamma_{21} \sqrt{\tau} \sum_{k=1}^{\infty} k\, p_{-k}\,;$$

- we have shown that $\sigma_{\rm tot} = \sigma[\dot{S} > 0] + \sigma[\dot{S} < 0] \geq 0\,;$
- define on this basis an efficiency ε, such that:
  - (i) $0 \leq \varepsilon \leq 1$;
  - (ii) ε is maximal at minimal total entropy production $\sigma_{\rm tot}$;
  - (iii) $\varepsilon \longrightarrow 1$ in the reversibility limit $\sigma_{\rm tot} \longrightarrow 0$.
- the simplest candidate is:

$$\varepsilon = \frac{-\sigma[\dot{S} < 0]}{\sigma[\dot{S} > 0]}\,;$$

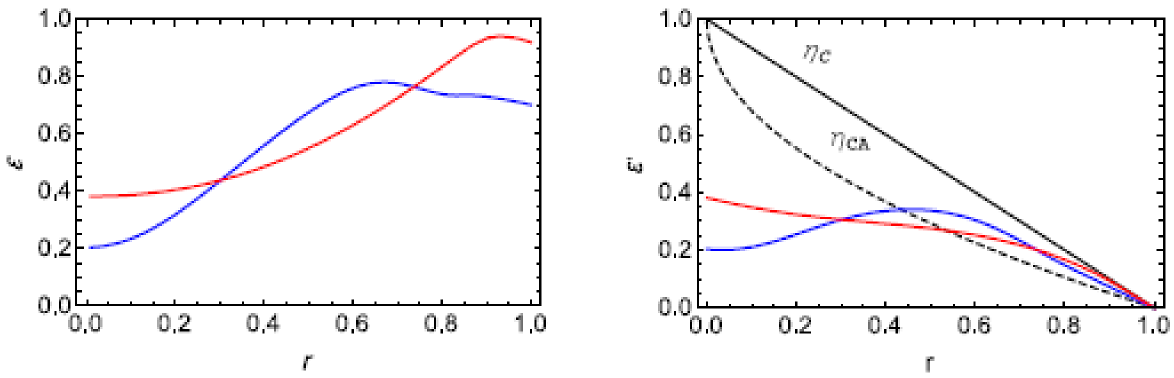
- in order to compare with η we introduce also

$$\varepsilon' = \frac{-\sigma[\dot{S} < 0]}{\sigma[\dot{S} > 0]}(1 - r) \leq \eta_C\,, \qquad r \equiv \frac{\beta_1}{\beta_2}\,, \qquad \beta_2 \geq \beta_1\,;$$

satisfying the Carnot bound by construction.

Plots illustrating ε and ε':

- analitycally ε and ε' are complicated (integration in ω), but numerics works well.



- typical behaviour of the efficiencies ε (left panel) and ε' (right panel):
  - for fermions - red line;
  - for bosons - blue line;
  - Carnot bound - black line;
  - $\eta_{CA} = 1 - \sqrt{r}$ is the Curzon-Ahlborn bound (endoreversible thermodynamics) - black dashed line;
- quantum fluctuations allow to exceed the CA bound;
- this is not the case in the mean value regime.

Summary and outlook:

- we developed a microscopic QFT approach to quantum transport based on the emission/absorption probabilities from the reservoirs;
- the proposed framework describes in a systematic way the quantum fluctuations at any order;
- we have seen that quantum systems of this type have two universal features:
  - transform heat to chemical energy or vice versa without dissipation;
  - produce entropy without explicit time reversal breaking.
- in this context we explored:
  - the fluctuation relations in a presence of defects;
  - a concept of efficiency beyond the meal value description;
- as expected, quantum statistics have a relevant impact in this context;

Non-equilibrium quantum thermodynamics - a new branch of quantum physics

Basic open question - properties of the **operator** $\dot{S}$ (quantum second law?).

Lesson from this investigation: study the probability distribution $\rho\,[\dot{S}]$: Test other models and non-equilibrium states.

- extend the above analysis to finite frequencies $\nu > 0$:
- experimental progress - Kolkowitz et al, Science (2015), Tikhonov et al, Nature Sci. Rep. (2016), Weng et al, Science (2018);
- partial theoretical progress - a bound state with energy $-\omega_b < 0$ has a

specific impact on the particle noise at frequency $\nu > \omega_b$;

bound state spectroscopy – Mintchev, Santoni, S. (2017);

- analyse more general domains $D = [a, b]$ and $D = \mathbb{R}$:
  - statistical interaction - anyon Tomonaga-Luttinger liquid:

    - quantum transport of anyon fluid in $\mathbb{R}$ – Mintchev, S. (2013).
  - Lieb-Liniger model in $\mathbb{R}$ (integrability): Calabrese et al. (2018) the

    distribution $\rho[\psi^*\psi]$ - turns out to be a Dirac comb;

    $\rho[\dot{S}]$ - still an open problem;

Many thanks for your attention