Fermionic quasibound states for Reissner-Nordsrtöm black holes

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Outline

- *This presentation is based on the paper: arXiv:1905.05086 to appear in Mod. Phys. Lett. A.*

- Introduction: Dirac eq. in curved spacetimes, Cartesian gauge
- Solutions to Dirac eq. in black hole geometries. Discrete energy spectrum
- Energy of the (quasi)bound states. Limiting cases
- Discussion of the results
- Conclusions

- The Dirac equation

$$\boxed{i\gamma^a D_a \psi - m\psi = 0}$$

it results from the gauge invariant action

$$\boxed{\mathcal{S} = \int \mathrm{d}^4x \sqrt{-g}\left\{\frac{i}{2}\overline{\psi}\gamma^a D_a\psi - \frac{i}{2}(\overline{D_a\psi})\gamma^a\psi - m\bar{\psi}\psi\right\}}$$

and can be written explicitly as

$$\boxed{\begin{aligned}&(i\gamma^a e_a^\mu \partial_\mu - m)\,\psi + \frac{i}{2}\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, e_a^\mu)\gamma^a\psi \\ &- e\gamma^a e_a^\mu A_\mu \psi - \frac{1}{4}\{\gamma^a, S^b{}_c\}\omega^{\ c}_{ab}\psi = 0\end{aligned}}$$

- Let us assume that a Dirac particle of mass m is moving freely (as a perturbation) in the central gravitational field of a spherically symmetric black hole of mass M with the line element

$$\boxed{ds^2 = h(r)dt^2 - \frac{dr^2}{h(r)} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)}$$

- For the Schwarzschild black hole:

$$\boxed{h(r) = 1 - \frac{2M}{r}}$$

- For the Reissner-Nordsrtöm black hole:

$$\boxed{h(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}}$$

- The tetrad fields $\hat{e}^a(x) = \hat{e}^a_\mu dx^\mu$ (i.e. the 1-forms) defining the Cartesian gauge are

$$
\begin{aligned}
\hat{e}^0 &= \sqrt{h(r)}dt \\
\hat{e}^1 &= \frac{1}{\sqrt{h(r)}}\sin\theta\cos\phi\, dr + r\,\cos\theta\cos\phi\, d\theta - r\,\sin\theta\sin\phi\, d\phi \\
\hat{e}^2 &= \frac{1}{\sqrt{h(r)}}\sin\theta\sin\phi\, dr + r\,\cos\theta\sin\phi\, d\theta + r\,\sin\theta\cos\phi\, d\phi \\
\hat{e}^3 &= \frac{1}{\sqrt{h(r)}}\cos\theta\, dr - r\,\sin\theta\, d\theta
\end{aligned}
$$

- In this gauge the Dirac equation is completely covariant under rotations

- Particle-like energy eigenspinors of positive frequency and energy E (I. I. Cotaescu, Mod. Phys. Lett. A 22, 2493, 2007)

$$\boxed{\begin{aligned}\psi(x) &= \psi_{E,j,m,\kappa}(t,r,\theta,\phi)\\ &= \frac{e^{-iEt}}{r\,h(r)^{1/4}}\left[f^{+}_{E,\kappa}(r)\Phi^{+}_{m_j,\kappa}(\theta,\phi)+f^{-}_{E,\kappa}(r)\Phi^{-}_{m_j,\kappa}(\theta,\phi)\right]\end{aligned}}$$

$f^{\pm}_{E,\kappa}(r)$ - radial wave functions.
$\Phi^{\pm}_{m_j,\kappa}(\theta,\phi)$ - usual four-component angular spinors.

- The antiparticle-like energy eigenspinors can be obtained directly using the charge conjugation as in the flat case:

$$\boxed{V_{E,j,m,\kappa} = (\psi_{E,j,m,\kappa})^{c} \equiv C(\bar{\psi}_{E,j,m,\kappa})^{T}, \quad C = i\gamma^2\gamma^0}$$

- The radial Dirac equation

$$\boxed{\begin{pmatrix} m\sqrt{h(r)}+V(r) & -h(r)\frac{d}{dr}+\frac{\kappa}{r}\sqrt{h(r)} \\ h(r)\frac{d}{dr}+\frac{\kappa}{r}\sqrt{h(r)} & -m\sqrt{h(r)}+V(r) \end{pmatrix}\begin{pmatrix} f^{+}(r) \\ f^{-}(r) \end{pmatrix}=E\begin{pmatrix} f^{+}(r) \\ f^{-}(r) \end{pmatrix}}$$

- The resulting radial problem cannot be solved analytically as it stays forcing one to resort to numerical methods (S. Dolan, et. al., Phys. Rev. D 74, 064005, 2006) or to some approximations (I. I. Cotaescu, Mod. Phys. Lett. A 22, 2493, 2007)
- introducing Novikov-like variables

$$\boxed{x=\sqrt{\frac{r}{r_{+}}-1}\in(0,\infty)}$$

for Schw. BH: $r_{+}=2M$ and for RN: $r_{+}=M+\sqrt{M^2-Q^2}$

- The radial wave functions solutions to the Dirac eq. (for the discrete energy spectrum, $\mu > \epsilon$) can be expressed in terms of Hypergeometric functions ${}_1F_1$.

$$\boxed{\begin{pmatrix} f_b^+(x) \\ f_b^-(x) \end{pmatrix} = \begin{pmatrix} -\sqrt{\mu+\varepsilon} & \sqrt{\mu+\varepsilon} \\ \sqrt{\mu-\varepsilon} & \sqrt{\mu-\varepsilon} \end{pmatrix} \begin{pmatrix} \hat{f}_b^+(x) \\ \hat{f}_b^-(x) \end{pmatrix}}$$

where $\mu = r_+ m$, $\varepsilon = r_+ E$ and

$$\hat{f}_b^+(x) = C^+ (2\nu)^{s+\frac{1}{2}} x^{2s} e^{-\nu x^2} \, {}_1F_1(s-p+1, 2s+1, 2\nu x^2)$$

$$\hat{f}_b^-(x) = C^- (2\nu)^{s+\frac{1}{2}} x^{2s} e^{-\nu x^2} \, {}_1F_1(s-p, 2s+1, 2\nu x^2)$$

- These solutions constitute the starting point for our study of the quasibound states in Schwarzschild and Reissner-Nordstrom black hole backgrounds.

- The normalization constants satisfy the conditions:

$$\frac{C^-}{C^+} = \frac{\nu(s+p)}{\kappa\nu + \beta\mu - \zeta\varepsilon}$$

and the parameters involved are

$$s = \sqrt{\kappa^2 + \zeta^2 - \beta^2}, \qquad p = \frac{\beta\varepsilon - \zeta\mu}{\nu}, \qquad \nu = \sqrt{\mu^2 - \varepsilon^2}$$

$$\zeta = \frac{1}{2}\mu(1 - \frac{r_-}{r_+}), \quad \beta = \varepsilon - eQ, \quad r_\pm = M \pm \sqrt{M^2 - Q^2}$$

Energy of the quasibound states

- The radial functions $\hat{f}_b^{\pm}(x)$

$$\boxed{\hat{f}_b^{-}(x) = C^{-}(2\nu)^{s+\frac{1}{2}} x^{2s} e^{-\nu x^2} \; {}_1F_1(s-p, 2s+1, 2\nu x^2)}$$

  are similar to the wave functions for a Dirac particle moving in a Coulomb potential.

- For obtaining the (quasi)bound state energies of fermions in the RN geometry we impose the standard quantization condition:

$$\boxed{s - p = -n_r} \qquad \boxed{\sqrt{\kappa^2 + \zeta^2 - \beta^2} - \frac{\beta\varepsilon - \zeta\mu}{\nu} = -n_r}$$

  with $n_r = 0, 1, 2, 3...$ the radial quantum number.

- The quantization condition $s - p = -n_r$ can be brought into the form:

$$\boxed{\frac{\varepsilon}{\mu} = \left[1 - \left(\frac{\beta\frac{\varepsilon}{\mu} - \zeta}{n_r + \sqrt{\kappa^2 + \zeta^2 - \beta^2}}\right)^2\right]^{\frac{1}{2}}}$$

- Assuming that the energy of the quasibound state is close to the rest energy of the fermion mc^2 and taking the limit $\varepsilon \to \mu$ we obtain:

$$\boxed{\frac{E}{mc^2} = \left[1 - \left(\frac{\mu - eQ - \zeta}{n_r + \sqrt{\kappa^2 + \zeta^2 - (\mu - eQ)^2}}\right)^2\right]^{\frac{1}{2}}}$$

that gives the energy of the quasibound state for a fermion in Reissner-Nordström geometry.

- The Schwarzschild result for the energy of a quasibound state (I. I. Cotaescu, Mod. Phys. Lett. A 22, 2493, 2007)

$$\frac{E}{mc^2} = \left[1 - \frac{\mu^2}{4\left(n_r + \sqrt{\kappa^2 - \frac{3}{4}\mu^2}\right)^2}\right]^{\frac{1}{2}}$$

is recovered by canceling the black hole's electric charge $Q = 0$.

- Taking the limit $M \to 0$ we obtain the discrete energy levels of the relativistic Dirac-Coulomb problem:

$$\frac{E}{mc^2} = \left[1 - \left(\frac{eQ}{n_r + \sqrt{\kappa^2 - (eQ)^2}}\right)^2\right]^{\frac{1}{2}}$$

$$\approx \left[1 + \frac{Z^2\alpha^2}{\left(n_r + \sqrt{\kappa^2 - Z^2\alpha^2}\right)^2}\right]^{-\frac{1}{2}}, \qquad Q = Ze = Z\sqrt{\alpha}$$

- The quantization condition eq. $s - p = -n_r$ can be solved analytically (for $n_r = 0$) or numerically (if $n_r \neq 0$).
- The energy of the ground state ($n_r = 0$) results to be:

$$\boxed{\frac{E_0}{mc^2} = \frac{mM \cdot eQ \pm \sqrt{\kappa^2\left[(mM)^2 + \kappa^2 - (eQ)^2\right]}}{(mM)^2 + \kappa^2}}$$

- By imposing the condition

$$\boxed{-\sqrt{\kappa^2 + (mM)^2} < eQ < \sqrt{\kappa^2 + (mM)^2}}$$

the ground state will always have a real energy, otherwise the energy of the state becomes complex.

- the states are labeled with the standard spectroscopic notation nL_j.
- the energy state $1S_{1/2}$ corresponds to the set of quantum numbers $(n = 1, l = 0, j = 1/2, \kappa = -1)$

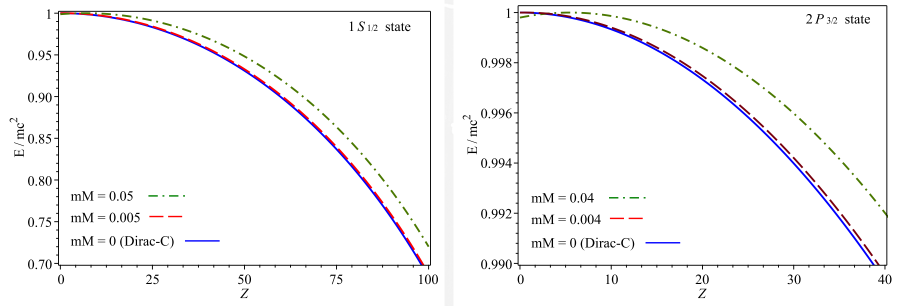

Figure: Comparison of the Reissner-Nordström ground state energy with the relativistic Dirac-Coulomb energy for the $1S_{1/2}$ state (left panel), respectively for the $2P_{3/2}$ state (right panel).

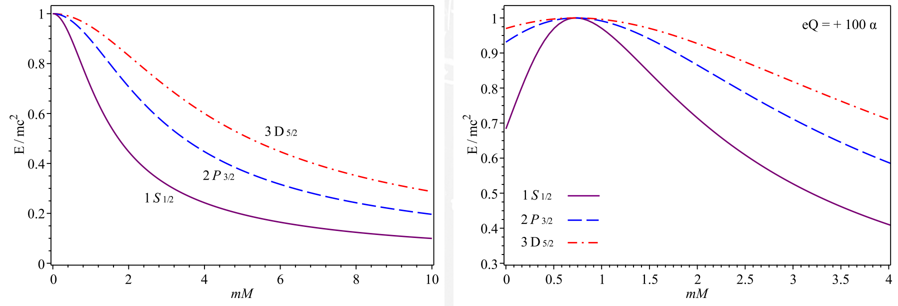

Figure: The energy spectra for Schwarzschild quasibound states (left panel), respectively for Reissner-Nordström quasibound states (right panel) as functions of the gravitational coupling $\alpha_g = \frac{mMG}{\hbar c}$ for states with $n_r = 0$.

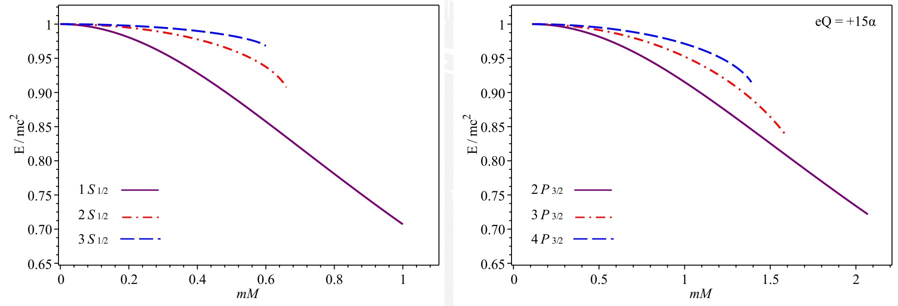

Figure: The energy spectra for Schwarzschild (*left panel*) and Reissner-Nordström (*right panel*) quasibound states as a function of the gravitational coupling $\alpha_g = \frac{mMG}{\hbar c}$ for states with $n_r \neq 0$.

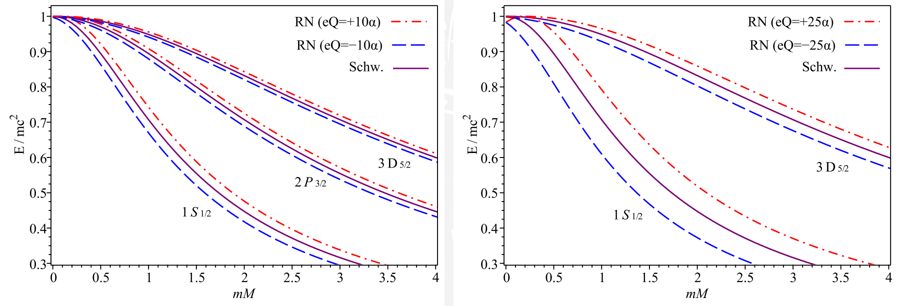

Figure: Comparison between the real part of the energy spectra of a Schwarzschild and Reissner-Nordström $1S_{1/2}$, $2P_{3/2}$, $3D_{5/2}$ quasibound states for: $eQ = \pm 10\alpha$ (*left panel*), respectively for $eQ = \pm 25\alpha$ (*right panel*). We observe that the spectra of RN quasibound states compared with the Schwarzschild one, is higher if the fermion and the black hole have the same type of charge (i.e. $eQ > 0$) (*left panel*), respectively is lower for the opposite case for which $eQ < 0$ (*right panel*).

Energy of the quasibound states

Discussion of the results

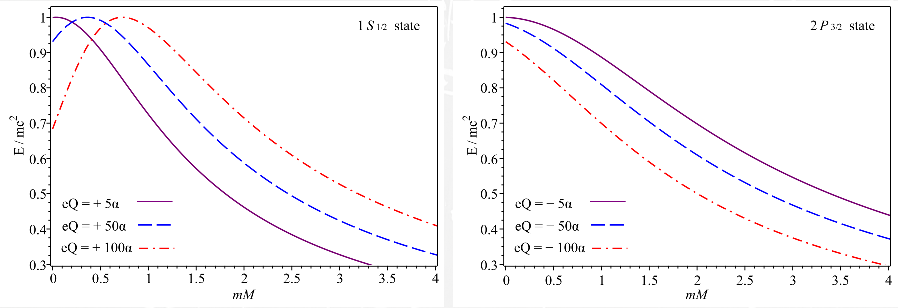

Figure: The energy spectra of the $1S_{1/2}$ quasibound state for a Reissner-Nordström black hole charged with positive or negative charges. In the left panel we observe that as eQ increases the spectra can have the same energy at two different values of the gravitational coupling $\alpha_g = mMG/\hbar c$.

Conclusions

- Obtaining the discrete quantum modes of the Dirac equation in a Reissner-Nordström background;
- Obtaining the (quasi)bound states of the Dirac field in Reissner-Nordström black hole geometry;
- Finding an analytical expression for the energy of the ground state;
- Reissner-Nordström quasibound states have higher energies compared with the Schwarzschild quasibound states if the black hole and the fermion have chargees with the same sign, otherwise the energy of the state is lower.;

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Thank you for your attention!

to infinity and beyond..

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