Chiral asymmetry in the weak interaction

Some applications of Clifford Algebras to the Standard Model

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Intro

Discrete properties of leptons and quarks:

Particle		e^-	$\overline{u}$	d	$\overline{\nu}$	ν	$\overline{d}$	u	e^+
Electric charge		-1	$-\frac{2}{3}$	$-\frac{1}{3}$	0	0	$+\frac{1}{3}$	$+\frac{2}{3}$	$+1$
Weak isospin	L	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$+\frac{1}{2}$	0	$+\frac{1}{2}$	0
	R	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$+\frac{1}{2}$	0	$+\frac{1}{2}$
Hypercharge	L	-1	$-\frac{4}{3}$	$+\frac{1}{3}$	0	-1	$+\frac{2}{3}$	$+\frac{1}{3}$	$+2$
	R	-2	$-\frac{1}{3}$	$-\frac{2}{3}$	1	0	$-\frac{1}{3}$	$+\frac{4}{3}$	$+1$

They are due to certain symmetry groups and representations.

Why these groups, and why these representations?

Intro

This can be partially explained by *grand unified theories* (GUT) like SU(5) (Georgi and Glashow 1974) and SO(10) (actually $\mathrm{Spin}(10)$) (Georgi 1975; Fritzsch and Minkowski 1975).

All such models based on a simple gauge group predict still undetected additional interactions and proton decay.

But they contain important insights for future developments.

Intro

But what if there is a simple mathematical structure which

- has the symmetries of the Standard Model,
- provides the right representations automatically,
- and predicts no additional particles and forces?

Result

		$\mathbf{1}_c$	$\mathbf{3}_c$			$\overline{\mathbf{1}}_c$	$\overline{\mathbf{3}}_c$		
Dirac, Lorentz	$\mathbf{1}_w$	ν_{R1}	u^r_{R1}	u^y_{R1}	u^b_{R1}	$\overline{e}_{L1}$	$\overline{d}^r_{L1}$	$\overline{d}^y_{L1}$	$\overline{d}^b_{L1}$
		ν_{R2}	u^r_{R2}	u^y_{R2}	u^b_{R2}	$\overline{e}_{L2}$	$\overline{d}^r_{L2}$	$\overline{d}^y_{L2}$	$\overline{d}^b_{L2}$
	$\mathbf{2}_w$	ν_{L1}	u^r_{L1}	u^y_{L1}	u^b_{L1}	$\overline{e}_{R1}$	$\overline{d}^r_{R1}$	$\overline{d}^y_{R1}$	$\overline{d}^b_{R1}$
		ν_{L2}	u^r_{L2}	u^y_{L2}	u^b_{L2}	$\overline{e}_{R2}$	$\overline{d}^r_{R2}$	$\overline{d}^y_{R2}$	$\overline{d}^b_{R2}$
Dirac, Lorentz		e_{L1}	d^r_{L1}	d^y_{L1}	d^b_{L1}	$\overline{\nu}_{R1}$	$\overline{u}^r_{R1}$	$\overline{u}^y_{R1}$	$\overline{u}^b_{R1}$
		e_{L2}	d^r_{L2}	d^y_{L2}	d^b_{L2}	$\overline{\nu}_{R2}$	$\overline{u}^r_{R2}$	$\overline{u}^y_{R2}$	$\overline{u}^b_{R2}$
	$\mathbf{1}_w$	e_{R1}	d^r_{R1}	d^y_{R1}	d^b_{R1}	$\overline{\nu}_{L1}$	$\overline{u}^r_{L1}$	$\overline{u}^y_{L1}$	$\overline{u}^b_{L1}$
		e_{R2}	d^r_{R2}	d^y_{R2}	d^b_{R2}	$\overline{\nu}_{L2}$	$\overline{u}^r_{L2}$	$\overline{u}^y_{L2}$	$\overline{u}^b_{L2}$

A three-dimensional Hermitian space χ determines a Clifford algebra $\mathbb{C}\ell\left(\chi^\dagger \oplus \chi\right)$, which is naturally split into left ideals.

Result

		$\mathbf{1}_c$	$\mathbf{3}_c$			$\overline{\mathbf{1}}_c$	$\overline{\mathbf{3}}_c$		
Dirac, Lorentz	$\mathbf{1}_w$	ν_{R1}	u^r_{R1}	u^y_{R1}	u^b_{R1}	$\overline{e}_{L1}$	$\overline{d}^r_{L1}$	$\overline{d}^y_{L1}$	$\overline{d}^b_{L1}$
		ν_{R2}	u^r_{R2}	u^y_{R2}	u^b_{R2}	$\overline{e}_{L2}$	$\overline{d}^r_{L2}$	$\overline{d}^y_{L2}$	$\overline{d}^b_{L2}$
	$\mathbf{2}_w$	ν_{L1}	u^r_{L1}	u^y_{L1}	u^b_{L1}	$\overline{e}_{R1}$	$\overline{d}^r_{R1}$	$\overline{d}^y_{R1}$	$\overline{d}^b_{R1}$
		ν_{L2}	u^r_{L2}	u^y_{L2}	u^b_{L2}	$\overline{e}_{R2}$	$\overline{d}^r_{R2}$	$\overline{d}^y_{R2}$	$\overline{d}^b_{R2}$
Dirac, Lorentz		e_{L1}	d^r_{L1}	d^y_{L1}	d^b_{L1}	$\overline{\nu}_{R1}$	$\overline{u}^r_{R1}$	$\overline{u}^y_{R1}$	$\overline{u}^b_{R1}$
		e_{L2}	d^r_{L2}	d^y_{L2}	d^b_{L2}	$\overline{\nu}_{R2}$	$\overline{u}^r_{R2}$	$\overline{u}^y_{R2}$	$\overline{u}^b_{R2}$
	$\mathbf{1}_w$	e_{R1}	d^r_{R1}	d^y_{R1}	d^b_{R1}	$\overline{\nu}_{L1}$	$\overline{u}^r_{L1}$	$\overline{u}^y_{L1}$	$\overline{u}^b_{L1}$
		e_{R2}	d^r_{R2}	d^y_{R2}	d^b_{R2}	$\overline{\nu}_{L2}$	$\overline{u}^r_{L2}$	$\overline{u}^y_{L2}$	$\overline{u}^b_{L2}$

In a basis adapted to the ideal decomposition, each column contains two 4-spinors associated to different flavors.

Result

		$\mathbf{1}_c$	$\mathbf{3}_c$			$\overline{\mathbf{1}}_c$	$\overline{\mathbf{3}}_c$		
Dirac, Lorentz	$\mathbf{1}_w$	ν_{R1}	u^r_{R1}	u^y_{R1}	u^b_{R1}	$\bar{e}_{L1}$	$\bar{d}^r_{L1}$	$\bar{d}^y_{L1}$	$\bar{d}^b_{L1}$
		ν_{R2}	u^r_{R2}	u^y_{R2}	u^b_{R2}	$\bar{e}_{L2}$	$\bar{d}^r_{L2}$	$\bar{d}^y_{L2}$	$\bar{d}^b_{L2}$
	$\mathbf{2}_w$	ν_{L1}	u^r_{L1}	u^y_{L1}	u^b_{L1}	$\bar{e}_{R1}$	$\bar{d}^r_{R1}$	$\bar{d}^y_{R1}$	$\bar{d}^b_{R1}$
		ν_{L2}	u^r_{L2}	u^y_{L2}	u^b_{L2}	$\bar{e}_{R2}$	$\bar{d}^r_{R2}$	$\bar{d}^y_{R2}$	$\bar{d}^b_{R2}$
Dirac, Lorentz		e_{L1}	d^r_{L1}	d^y_{L1}	d^b_{L1}	$\bar{\nu}_{R1}$	$\bar{u}^r_{R1}$	$\bar{u}^y_{R1}$	$\bar{u}^b_{R1}$
		e_{L2}	d^r_{L2}	d^y_{L2}	d^b_{L2}	$\bar{\nu}_{R2}$	$\bar{u}^r_{R2}$	$\bar{u}^y_{R2}$	$\bar{u}^b_{R2}$
	$\mathbf{1}_w$	e_{R1}	d^r_{R1}	d^y_{R1}	d^b_{R1}	$\bar{\nu}_{L1}$	$\bar{u}^r_{L1}$	$\bar{u}^y_{L1}$	$\bar{u}^b_{L1}$
		e_{R2}	d^r_{R2}	d^y_{R2}	d^b_{R2}	$\bar{\nu}_{L2}$	$\bar{u}^r_{L2}$	$\bar{u}^y_{L2}$	$\bar{u}^b_{L2}$

The Lie group $SU(3)_c$ permutes the columns according to the representations $\mathbf{1}_c$, $\mathbf{3}_c$, $\overline{\mathbf{1}}_c$, and $\overline{\mathbf{3}}_c$.

Result

		$\mathbf{1}_c$	$\mathbf{3}_c$			$\overline{\mathbf{1}}_c$	$\overline{\mathbf{3}}_c$		
Dirac, Lorentz	$\mathbf{1}_w$	ν_{R1}	u^r_{R1}	u^y_{R1}	u^b_{R1}	$\bar{e}_{L1}$	$\bar{d}^r_{L1}$	$\bar{d}^y_{L1}$	$\bar{d}^b_{L1}$
		ν_{R2}	u^r_{R2}	u^y_{R2}	u^b_{R2}	$\bar{e}_{L2}$	$\bar{d}^r_{L2}$	$\bar{d}^y_{L2}$	$\bar{d}^b_{L2}$
	$\mathbf{2}_w$	ν_{L1}	u^r_{L1}	u^y_{L1}	u^b_{L1}	$\bar{e}_{R1}$	$\bar{d}^r_{R1}$	$\bar{d}^y_{R1}$	$\bar{d}^b_{R1}$
		ν_{L2}	u^r_{L2}	u^y_{L2}	u^b_{L2}	$\bar{e}_{R2}$	$\bar{d}^r_{R2}$	$\bar{d}^y_{R2}$	$\bar{d}^b_{R2}$
Dirac, Lorentz		e_{L1}	d^r_{L1}	d^y_{L1}	d^b_{L1}	$\bar{\nu}_{R1}$	$\bar{u}^r_{R1}$	$\bar{u}^y_{R1}$	$\bar{u}^b_{R1}$
		e_{L2}	d^r_{L2}	d^y_{L2}	d^b_{L2}	$\bar{\nu}_{R2}$	$\bar{u}^r_{R2}$	$\bar{u}^y_{R2}$	$\bar{u}^b_{R2}$
	$\mathbf{1}_w$	e_{R1}	d^r_{R1}	d^y_{R1}	d^b_{R1}	$\bar{\nu}_{L1}$	$\bar{u}^r_{L1}$	$\bar{u}^y_{L1}$	$\bar{u}^b_{L1}$
		e_{R2}	d^r_{R2}	d^y_{R2}	d^b_{R2}	$\bar{\nu}_{L2}$	$\bar{u}^r_{L2}$	$\bar{u}^y_{L2}$	$\bar{u}^b_{L2}$

Each ideal is indexed with an electric charge which is multiple of $\frac{1}{3}$ partially representing the charge of the upper particle,

Result

		$\mathbf{1}_c$	$\mathbf{3}_c$			$\overline{\mathbf{1}}_c$	$\overline{\mathbf{3}}_c$		
Dirac, Lorentz	$\mathbf{1}_w$	ν_{R1}	u^r_{R1}	u^y_{R1}	u^b_{R1}	$\bar{e}_{L1}$	$\bar{d}^r_{L1}$	$\bar{d}^y_{L1}$	$\bar{d}^b_{L1}$
		ν_{R2}	u^r_{R2}	u^y_{R2}	u^b_{R2}	$\bar{e}_{L2}$	$\bar{d}^r_{L2}$	$\bar{d}^y_{L2}$	$\bar{d}^b_{L2}$
	$\mathbf{2}_w$	ν_{L1}	u^r_{L1}	u^y_{L1}	u^b_{L1}	$\bar{e}_{R1}$	$\bar{d}^r_{R1}$	$\bar{d}^y_{R1}$	$\bar{d}^b_{R1}$
		ν_{L2}	u^r_{L2}	u^y_{L2}	u^b_{L2}	$\bar{e}_{R2}$	$\bar{d}^r_{R2}$	$\bar{d}^y_{R2}$	$\bar{d}^b_{R2}$
Dirac, Lorentz		e_{L1}	d^r_{L1}	d^y_{L1}	d^b_{L1}	$\bar{\nu}_{R1}$	$\bar{u}^r_{R1}$	$\bar{u}^y_{R1}$	$\bar{u}^b_{R1}$
		e_{L2}	d^r_{L2}	d^y_{L2}	d^b_{L2}	$\bar{\nu}_{R2}$	$\bar{u}^r_{R2}$	$\bar{u}^y_{R2}$	$\bar{u}^b_{R2}$
	$\mathbf{1}_w$	e_{R1}	d^r_{R1}	d^y_{R1}	d^b_{R1}	$\bar{\nu}_{L1}$	$\bar{u}^r_{L1}$	$\bar{u}^y_{L1}$	$\bar{u}^b_{L1}$
		e_{R2}	d^r_{R2}	d^y_{R2}	d^b_{R2}	$\bar{\nu}_{L2}$	$\bar{u}^r_{L2}$	$\bar{u}^y_{L2}$	$\bar{u}^b_{L2}$

and its color is determined by the ideal to which belongs, having associated a particular representation of $SU(3)_c$.

Result

		$\mathbf{1}_c$	$\mathbf{3}_c$			$\overline{\mathbf{1}}_c$	$\overline{\mathbf{3}}_c$		
Dirac, Lorentz	$\mathbf{1}_w$	ν_{R1}	u^r_{R1}	u^y_{R1}	u^b_{R1}	$\overline{e}_{L1}$	$\overline{d}^r_{L1}$	$\overline{d}^y_{L1}$	$\overline{d}^b_{L1}$
		ν_{R2}	u^r_{R2}	u^y_{R2}	u^b_{R2}	$\overline{e}_{L2}$	$\overline{d}^r_{L2}$	$\overline{d}^y_{L2}$	$\overline{d}^b_{L2}$
	$\mathbf{2}_w$	ν_{L1}	u^r_{L1}	u^y_{L1}	u^b_{L1}	$\overline{e}_{R1}$	$\overline{d}^r_{R1}$	$\overline{d}^y_{R1}$	$\overline{d}^b_{R1}$
		ν_{L2}	u^r_{L2}	u^y_{L2}	u^b_{L2}	$\overline{e}_{R2}$	$\overline{d}^r_{R2}$	$\overline{d}^y_{R2}$	$\overline{d}^b_{R2}$
Dirac, Lorentz		e_{L1}	d^r_{L1}	d^y_{L1}	d^b_{L1}	$\overline{\nu}_{R1}$	$\overline{u}^r_{R1}$	$\overline{u}^y_{R1}$	$\overline{u}^b_{R1}$
		e_{L2}	d^r_{L2}	d^y_{L2}	d^b_{L2}	$\overline{\nu}_{R2}$	$\overline{u}^r_{R2}$	$\overline{u}^y_{R2}$	$\overline{u}^b_{R2}$
	$\mathbf{1}_w$	e_{R1}	d^r_{R1}	d^y_{R1}	d^b_{R1}	$\overline{\nu}_{L1}$	$\overline{u}^r_{L1}$	$\overline{u}^y_{L1}$	$\overline{u}^b_{L1}$
		e_{R2}	d^r_{R2}	d^y_{R2}	d^b_{R2}	$\overline{\nu}_{L2}$	$\overline{u}^r_{L2}$	$\overline{u}^y_{L2}$	$\overline{u}^b_{L2}$

The actions of the Dirac algebra and the Lorentz group are reducible, permute the rows of each ideal, and split it naturally into two 4-spinors, whose left chiral components are permuted by the representations $\mathbf{1}_w$ and $\mathbf{2}_w$ of $SU(2)_L$.

Exterior algebra patterns in the Standard Model

Exterior algebra patterns – Weak symmetry $\mathrm{SU}(2)_\mathrm{L}$

The standard representation of $\mathrm{SU}(n)$ is $\mathbb{C}_n \cong \mathbb{C}^n$.

The fundamental and the trivial representations of $\mathrm{SU}(n)$ are $\bigwedge^k \mathbb{C}_n$.

The representations of the weak interaction group $\mathrm{SU}(2)_\mathrm{L}$

Representation	Particles	Hypercharge
$\bigwedge^0 \mathbb{C}_2$	$(\nu_e)_R$ **(?)**	0
$\bigwedge^1 \mathbb{C}_2$	$(\nu_e, e^-)_L$	-1
$\bigwedge^2 \mathbb{C}_2$	$(e^-)_R$	-2

(?) Does the right handed neutrino exist?

Exterior algebra patterns – Color symmetry $\mathrm{SU}(3)_c$

Combined internal charge and color spaces:

Representation	Particles	Electric charge
$\bigwedge^0 \mathbb{C}_3$	$\overline{\nu}_e$	0
$\bigwedge^1 \mathbb{C}_3$	d^r, d^y, d^b	$-\frac{1}{3}$
$\bigwedge^2 \mathbb{C}_3$	$\overline{u}^{\overline{r}}, \overline{u}^{\overline{y}}, \overline{u}^{\overline{b}}$	$-\frac{2}{3}$
$\bigwedge^3 \mathbb{C}_3$	e^-	-1

Representations of complex Clifford algebras, dim $2r$ i

The classification of complex Clifford algebras says $\mathbb{C}\ell_{2r} \cong \mathbb{C}(2^r)$. To see this, consider the orthonormal basis $(e_1, \ldots, e_r, e_{r+1}, \ldots, e_{2r})$, where $e_j^2 = 1$.

Then, we can build the basis

$$\begin{cases} a_j := \frac{1}{2}(e_j + ie_{r+j}) \\ a_j^\dagger := \frac{1}{2}(e_j - ie_{r+j}) \end{cases} \tag{1}$$

Then,

$$\begin{cases} \{a_j, a_k\} = \{a_j^\dagger, a_k^\dagger\} = 0 \\ \{a_j, a_k^\dagger\} = \delta_{jk}. \end{cases} \tag{2}$$

This is a *Witt basis*.

Representations of complex Clifford algebras, dim $2r$ ii

The *Witt decomposition* of the vector space V is $V := W \oplus W^\dagger$, where W is spanned by (a_j), and W is spanned by $(a_j^\dagger)$.

Let $a \in \bigwedge^r W$, $a := a_1 \wedge \ldots \wedge a_r = a_1 \ldots a_r$. Then, a is *nilpotent* ($a^2 = 0$), so $\bigwedge^\bullet W^\dagger a$ is a left ideal.

On the space $\bigwedge^\bullet W^\dagger a$, $a^\dagger$ and a act like *creation* and *annihilation* operators. Let $\phi \in \bigwedge^\bullet W^\dagger$. Then,

$$\begin{cases} a_j^\dagger \phi a = a_j^\dagger \wedge \phi a \\ a_j \phi a = i_{a_j} \phi a. \end{cases} \tag{3}$$

Since $\dim W^\dagger = 2^r$, this is our irreducible representation of $\mathbb{C}\ell_{2r}$.

Algebraic spinors of $\mathbb{C}\ell_{2r}$

Since on W the inner product vanishes, the Clifford product on the subalgebra generated by W coincides with the exterior product. The same holds for $W^\dagger$.

The algebra $\mathbb{C}\ell_{2r}$ is spanned by elements of the form

$$a_{j_1}^\dagger \dots a_{j_p}^\dagger a a^\dagger a_{k_1} \dots a_{k_q}, \tag{4}$$

$p, q \in \{0, \dots, r\}$, $1 \leq j_1 < \dots < j_p \leq r$, $1 \leq k_1 < \dots < k_q \leq r$.

Since the elements of the form $a_{j_1}^\dagger \dots a_{j_p}^\dagger$ span $\bigwedge^\bullet W^\dagger$, $\mathbb{C}\ell_{2r}$ is the direct sum of the minimal left ideals of the form

$$\bigwedge{}^\bullet W^\dagger a a^\dagger a_{k_1} \dots a_{k_q}. \tag{5}$$

On these ideals, $\mathbb{C}\ell_{2r}$ is represented just like on $\bigwedge^\bullet W^\dagger a$ (3).

From exterior algebra to Clifford algebra $\mathbb{C}\ell_{2r}$

Conversely, we can start with $\mathbb{C}_r$ endowed with a Hermitian inner product $\mathfrak{h}$. Then, there is an isomorhism $\overline{\mathbb{C}_r} \cong \mathbb{C}_r^*$, due to $\mathfrak{h}$.

Let $u, v \in \bigwedge^\bullet \mathbb{C}_r$. On $\bigwedge^\bullet(\overline{\mathbb{C}_r} \oplus \mathbb{C}_r) \cong \bigwedge^\bullet \overline{\mathbb{C}_r} \otimes_{\mathbb{C}} \bigwedge^\bullet \mathbb{C}_r$, define an associative product by $uv := u \wedge v$, $u^\dagger v^\dagger := u^\dagger \wedge v^\dagger$, $uv^\dagger = u \wedge v^\dagger + \frac{1}{2} v^\dagger(u)$, $u^\dagger v = u^\dagger \wedge v + \frac{1}{2} u^\dagger(v)$.

Then, if (a_j) is an orthonormal basis of $\mathbb{C}_r$,

$$\begin{cases} \{a_j, a_k\} = \{a_j^\dagger, a_k^\dagger\} = 0 \\ \{a_j, a_k^\dagger\} = \delta_{jk}. \end{cases} \tag{6}$$

One obtains a Clifford algebra $\mathbb{C}\ell(\overline{\mathbb{C}_r} \oplus \mathbb{C}_r) \cong \mathbb{C}\ell_{2r}$.

Note that we don't need $\mathfrak{h}$ if we start from $\bigwedge^\bullet(\mathbb{C}_r^* \oplus \mathbb{C}_r)$.

Exterior algebra patterns – Chiral spinors

Let γ^μ in the Dirac algebra $\mathbb{C}\ell_4 = C\ell_{1,3} \otimes \mathbb{C}$.

Define

$$\begin{cases} e_1 = \frac{1}{2}(\gamma_0 + \gamma_3), e_2 = \frac{1}{2}(-i\gamma_2 + \gamma_1) \\ f_1 = \frac{1}{2}(\gamma_0 - \gamma_3), f_2 = \frac{1}{2}(-i\gamma_2 - \gamma_1). \end{cases} \tag{7}$$

Then, $f_1 f_2$ is nilpotent, and defines a minimal left ideal $\mathbb{C}\ell_4 f_1 f_2$.

In the basis $(e_1 f_1 f_2, e_2 f_1 f_2, 1 f_1 f_2, e_1 e_2 f_1 f_2)$ of $\mathbb{C}\ell_4 f_1 f_2$, the matrix form of γ^μ is the Weyl representation.

Let Σ be spanned by (e_1, e_2). We see that the spinors from $\bigwedge^{-}\Sigma f_1 f_2$ are Weyl spinors of left chirality, and those from $\bigwedge^{+}\Sigma f_1 f_2$ are Weyl spinors of right chirality.

Exterior algebra patterns – Summary

1. Exterior algebras are the representations of the gauge groups in the Standard Model. The leptons and quarks in a generation correspond to these representations.
2. The classification of leptons and quarks and their antiparticles given by $\bigwedge^{\bullet}\mathbb{C}_3$ and $\bigwedge^{\bullet}\overline{\mathbb{C}_3}$ suggests $\mathbb{C}\ell_{3+3}$.
3. The weak force acts on the odd part $\bigwedge^{1}\mathbb{C}_2 = \bigwedge^{-}\mathbb{C}_2$ of $\bigwedge\mathbb{C}_2$. This suggests $\mathbb{C}\ell_{2+2}$. But since the same particles belong to representations of $\mathsf{SU}(3)_c$, this suggests that $\mathbb{C}\ell_{2+2}$ is a subalgebra of the $\mathbb{C}\ell_{3+3}$ algebra above. The odd Dirac spinors are the left handed spinors. The weak force acts on left handed spinors. These suggest that the relation between chirality and the weak force is due to the inclusion of both the Dirac algebra and $\mathbb{C}\ell_{2+2}$ in the same algebra, which is $\mathbb{C}\ell_{3+3}$.

SMA – Definition and main properties

Let us define $\chi := \overline{\chi}_{\mathrm{em}} \otimes \overline{\chi}_{\mathrm{c}}$. The space χ has complex dimension three and has the Hermitian inner product $\mathfrak{h} = h_{em} \otimes h_c$.

Orthonormal basis:

$$\begin{cases} (\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3) \\ (\mathfrak{q}^\dagger{}_1, \mathfrak{q}^\dagger{}_2, \mathfrak{q}^\dagger{}_3). \end{cases} \tag{8}$$

The combined internal charge and color spaces for fermions and leptons are represented now in this table:

Particle	$\mathbf{e}^-$	$\overline{\mathbf{u}}$	$\mathbf{d}$	$\overline{\nu}$	ν	$\overline{\mathbf{d}}$	$\mathbf{u}$	$\mathbf{e}^+$
Electro-color space	$\bigwedge^3\chi$	$\bigwedge^2\chi$	$\bigwedge^1\chi$	$\bigwedge^0\chi$	$\bigwedge^0\overline{\chi}$	$\bigwedge^1\overline{\chi}$	$\bigwedge^2\overline{\chi}$	$\bigwedge^3\overline{\chi}$

SMA – Definition and main properties ii

On the space $\chi^\dagger \oplus \chi$ we define the *inner product*

$$\langle u_1^\dagger + u_2, u_3^\dagger + u_4 \rangle := \frac{1}{2}\Big(u_1^\dagger(u_4) + u_3^\dagger(u_2)\Big) \in \mathbb{C}, \tag{9}$$

where $u_1^\dagger, u_3^\dagger \in \chi^\dagger$ and $u_2, u_4 \in \chi$ (also see Gualtieri 2004).

We call *Standard Model Algebra* (SMA) the Clifford algebra defined by the inner product (9),

$$\mathcal{A}_{\mathrm{SM}} := \mathbb{C}\ell(\chi^\dagger \oplus \chi) \cong \mathbb{C}\ell_6, \tag{10}$$

together with the *Witt decomposition* $\chi^\dagger \oplus \chi$ of the base complex 6-dimensional space, and with the Hermitian inner product on χ and $\chi^\dagger$.

SMA – Definition and main properties iii

The elements of the bases defined in equation (8) satisfy the anticommutation relations

$$\begin{aligned} \{\mathfrak{q}_j, \mathfrak{q}_k\} &= 0, \\ \{\mathfrak{q}^\dagger_j, \mathfrak{q}^\dagger_k\} &= 0, \\ \{\mathfrak{q}_j, \mathfrak{q}^\dagger_k\} &= \delta_{jk} \end{aligned} \tag{11}$$

for $j, k \in \{1, 2, 3\}$.

SMA – Definition and main properties iv

We define an orthonormal basis of the vector space $\chi^\dagger \oplus \chi$, $(\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \tilde{\mathfrak{e}}_1, \tilde{\mathfrak{e}}_2, \tilde{\mathfrak{e}}_3)$, by

$$\begin{cases} \mathfrak{e}_j = \mathfrak{q}_j + \mathfrak{q}_j^\dagger \\ \tilde{\mathfrak{e}}_j = i\left(\mathfrak{q}_j^\dagger - \mathfrak{q}_j\right), \end{cases} \tag{12}$$

where $j \in \{1, 2, 3\}$.

Then, $\mathfrak{e}_j^2 = 1$, $\tilde{\mathfrak{e}}_j^2 = 1$, $\mathfrak{e}_j^\dagger = \mathfrak{e}_j$, and $\tilde{\mathfrak{e}}_j^\dagger = \tilde{\mathfrak{e}}_j$. Also,

$$\begin{cases} \mathfrak{q}_j = \frac{1}{2}\left(\mathfrak{e}_j + i\tilde{\mathfrak{e}}_j\right) \\ \mathfrak{q}_j^\dagger = \frac{1}{2}\left(\mathfrak{e}_j - i\tilde{\mathfrak{e}}_j\right). \end{cases} \tag{13}$$

Ideals and representation

The elements

$$\begin{cases} \mathfrak{q} := & \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3, \\ \mathfrak{q}^\dagger = & \mathfrak{q}^\dagger_3\mathfrak{q}^\dagger_2\mathfrak{q}^\dagger_1, \end{cases} \tag{14}$$

are *nilpotent* ($\mathfrak{q}^2 = 0$ and $\mathfrak{q}^{\dagger 2} = 0$).

We make the notation $\mathfrak{p} := \mathfrak{q}\mathfrak{q}^\dagger$ and $\mathfrak{p}' = \mathfrak{q}^\dagger\mathfrak{q}$.

Then,

$$\mathfrak{p} = \frac{1 + i\mathfrak{e}_1\tilde{\mathfrak{e}}_1}{2} \cdot \frac{1 + i\mathfrak{e}_2\tilde{\mathfrak{e}}_2}{2} \cdot \frac{1 + i\mathfrak{e}_3\tilde{\mathfrak{e}}_3}{2}. \tag{15}$$

The elements $\mathfrak{p}$ and $\mathfrak{p}'$ are idempotent, $(\mathfrak{p})^2 = \mathfrak{p}$ and $(\mathfrak{p}')^2 = \mathfrak{p}'$.

SMA – Ideals

They are in fact primitive idempotent elements, hence they define minimal left and right ideals of the algebra $\mathcal{A}_{\mathsf{SM}}$ (Chevalley 1997; Crumeyrolle 1990). When we represent the Clifford algebra $\mathcal{A}_{\mathsf{SM}}$ as an endomorphism algebra $\mathsf{End}_{\mathbb{C}}(\mathbb{C}^8)$, the idempotents $\mathfrak{p}$ and $\mathfrak{p}'$ are represented as projectors.

The ideals $\mathcal{A}_{\mathsf{SM}}\mathfrak{q}^\dagger = \mathcal{A}_{\mathsf{SM}}\mathfrak{p}$ and $\mathcal{A}_{\mathsf{SM}}\mathfrak{q} = \mathcal{A}_{\mathsf{SM}}\mathfrak{p}'$ are minimal left ideals, and the ideals $\mathfrak{q}\mathcal{A}_{\mathsf{SM}} = (\mathcal{A}_{\mathsf{SM}}\mathfrak{q}^\dagger)^\dagger$ and $\mathfrak{q}^\dagger\mathcal{A}_{\mathsf{SM}} = (\mathcal{A}_{\mathsf{SM}}\mathfrak{q})^\dagger$ are minimal right ideals. It is easy to show that $\bigwedge^\bullet\chi\mathfrak{q} = 0$ and $\bigwedge^\bullet\chi^\dagger\mathfrak{q}^\dagger = 0$, and therefore $\mathcal{A}_{\mathsf{SM}}\mathfrak{q}^\dagger = \bigwedge^\bullet\chi\mathfrak{q}^\dagger = \bigwedge^\bullet\chi^\dagger\mathfrak{p}$ and $\mathcal{A}_{\mathsf{SM}}\mathfrak{q} = \bigwedge^\bullet\chi^\dagger\mathfrak{q} = \bigwedge^\bullet\chi\mathfrak{p}'$. Similar relations hold for the minimal right ideals, $\mathfrak{q}\bigwedge^\bullet\chi = 0$, $\mathfrak{q}^\dagger\bigwedge^\bullet\chi^\dagger = 0$, $\mathfrak{q}\mathcal{A}_{\mathsf{SM}} = \mathfrak{q}\bigwedge^\bullet\chi^\dagger = \mathfrak{p}\bigwedge^\bullet\chi$, and $\mathfrak{q}^\dagger\mathcal{A}_{\mathsf{SM}} = \mathfrak{q}^\dagger\bigwedge^\bullet\chi = \mathfrak{p}'\bigwedge^\bullet\chi^\dagger$.

Representation of $\mathcal{A}_{\mathrm{SM}}$ on its ideal $\bigwedge^\bullet \chi^\dagger \mathfrak{p}$

The Clifford product between $u^\dagger + v \in \chi^\dagger \oplus \chi$ and $\omega\mathfrak{p} \in \bigwedge^\bullet \chi^\dagger \mathfrak{p}$ is

$$(u^\dagger + v)\omega\mathfrak{p} = (u^\dagger \wedge \omega)\mathfrak{p} + (i_v\omega)\mathfrak{p} \in \bigwedge{}^\bullet \chi^\dagger \mathfrak{p}, \tag{16}$$

where the *interior product* $i_v\omega$ is defined for any $\omega \in \bigwedge^k \chi^\dagger$ by

$$(i_v\omega)(u_1, \ldots, u_k) = \begin{cases} \omega(v, u_1, \ldots, u_{k-1}), & \text{for } k \in \{1,2,3\}, \text{ and} \\ 0 & \text{for } k = 0. \end{cases} \tag{17}$$

Then, $\mathfrak{q}_j$ and $\mathfrak{q}_j^\dagger$ act as *ladder operators* on $\bigwedge^\bullet \chi^\dagger \mathfrak{p}$:

$$\begin{cases} \mathfrak{q}_j^\dagger(\omega\mathfrak{p}) = (\mathfrak{q}_j^\dagger \wedge \omega)\mathfrak{p}, \\ \mathfrak{q}_j(\omega\mathfrak{p}) = (i_{\mathfrak{q}_j}\omega)\mathfrak{p}, \end{cases} \tag{18}$$

which is consistent with the anticommutation relations (11).

Similarly to equation (16) one defines an irreducible representation on the minimal left ideal $\bigwedge^\bullet \chi \mathfrak{p}'$ of $\mathcal{A}_{\mathrm{SM}}$. Its elements are $\mathbb{C}\ell_6$-spinors.

Representation of $\mathcal{A}_{\mathsf{SM}}$ on its ideal $\bigwedge{}^\bullet\chi^\dagger\mathfrak{p}$

A basis of the ideal $\bigwedge{}^\bullet\chi^\dagger\mathfrak{p}$ is

$$(1\ \mathfrak{p}, \mathfrak{q}^\dagger_{23}\ \mathfrak{p}, \mathfrak{q}^\dagger_{31}\ \mathfrak{p}, \mathfrak{q}^\dagger_{12}\ \mathfrak{p}, \mathfrak{q}^\dagger_{321}\ \mathfrak{p}, \mathfrak{q}^\dagger_{1}\ \mathfrak{p}, \mathfrak{q}^\dagger_{2}\ \mathfrak{p}, \mathfrak{q}^\dagger_{3}\ \mathfrak{p}). \tag{19}$$

The basis (19) is written in terms of the idempotent element $\mathfrak{p}$. It is equal to the basis

$$(\mathfrak{q}\ \mathfrak{q}^\dagger, -\mathfrak{q}_1\ \mathfrak{q}^\dagger, -\mathfrak{q}_2\ \mathfrak{q}^\dagger, -\mathfrak{q}_3\ \mathfrak{q}^\dagger, 1\ \mathfrak{q}^\dagger, \mathfrak{q}_{23}\ \mathfrak{q}^\dagger, \mathfrak{q}_{31}\ \mathfrak{q}^\dagger, \mathfrak{q}_{12}\ \mathfrak{q}^\dagger) \tag{20}$$

written in terms of the nilpotent $\mathfrak{q}^\dagger$, which determines the same ideal as $\mathfrak{p}$.

Matrix representation of $\mathcal{A}_{\mathsf{SM}}$ on its ideal $\bigwedge^\bullet \chi^\dagger \mathfrak{p}$

Let us find the matrix representation of $\mathfrak{q}_j$, $\mathfrak{q}_j^\dagger$, $\mathfrak{e}_j$, and $\tilde{\mathfrak{e}}_j$ in the basis (19).

Here and in other places it will be convenient to use the *Pauli matrices* $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and the matrices $\sigma_+ = \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma_- = \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\sigma_3^+ = \frac{1}{2}(1+\sigma_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \sigma_+\sigma_-$, and $\sigma_3^- = \frac{1}{2}(1-\sigma_3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \sigma_-\sigma_+$.

Matrix representation of $\mathcal{A}_{\mathsf{SM}}$ on its ideal $\bigwedge{}^\bullet \chi^\dagger \mathfrak{p}$

We obtain, in the representation (19) of $\mathcal{A}_{\mathsf{SM}}$ on its ideal $\bigwedge{}^\bullet \chi^\dagger \mathfrak{q}$,

$$\mathfrak{q}^\dagger{}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\sigma_2 \\ -i\sigma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{q}_1 = \begin{pmatrix} 0 & 0 & i\sigma_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i\sigma_2 & 0 & 0 \end{pmatrix}, \tag{21}$$

$$\mathfrak{q}^\dagger{}_2 = \begin{pmatrix} 0 & 0 & 0 & \sigma_3^- \\ 0 & 0 & -\sigma_3^- & 0 \\ 0 & -\sigma_3^+ & 0 & 0 \\ \sigma_3^+ & 0 & 0 & 0 \end{pmatrix}, \mathfrak{q}_2 = \begin{pmatrix} 0 & 0 & 0 & \sigma_3^+ \\ 0 & 0 & -\sigma_3^+ & 0 \\ 0 & -\sigma_3^- & 0 & 0 \\ \sigma_3^- & 0 & 0 & 0 \end{pmatrix}, \tag{22}$$

Matrix representation of $\mathcal{A}_{\mathrm{SM}}$ on its ideal $\bigwedge^\bullet \chi^\dagger \mathfrak{p}$

$$\mathfrak{q}^\dagger_3 = \begin{pmatrix} 0 & 0 & 0 & -\sigma_- \\ 0 & 0 & \sigma_+ & 0 \\ 0 & -\sigma_+ & 0 & 0 \\ \sigma_- & 0 & 0 & 0 \end{pmatrix}, \mathfrak{q}_3 = \begin{pmatrix} 0 & 0 & 0 & \sigma_+ \\ 0 & 0 & -\sigma_- & 0 \\ 0 & \sigma_- & 0 & 0 \\ -\sigma_+ & 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

Then,

$$\mathfrak{q}^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sigma_3^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{q} = \begin{pmatrix} 0 & 0 & \sigma_3^+ & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{24}$$

$$\mathfrak{p} = \begin{pmatrix} \sigma_3^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{p}' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3^+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{25}$$

Matrix representation of $\mathcal{A}_{\mathrm{SM}}$ on its ideal $\bigwedge^\bullet \chi^\dagger \mathfrak{p}$

Then, from equation (12),

$$\mathfrak{e}_1 = \begin{pmatrix} 0 & 0 & i\sigma_2 & 0 \\ 0 & 0 & 0 & -i\sigma_2 \\ -i\sigma_2 & 0 & 0 & 0 \\ 0 & i\sigma_2 & 0 & 0 \end{pmatrix}, \tilde{\mathfrak{e}}_1 = \begin{pmatrix} 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & 0 & \sigma_2 \\ \sigma_2 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \end{pmatrix}, \tag{26}$$

$$\mathfrak{e}_2 = \begin{pmatrix} 0 & 0 & 0 & 1_2 \\ 0 & 0 & -1_2 & 0 \\ 0 & -1_2 & 0 & 0 \\ 1_2 & 0 & 0 & 0 \end{pmatrix}, \tilde{\mathfrak{e}}_2 = \begin{pmatrix} 0 & 0 & 0 & -i\sigma_3 \\ 0 & 0 & i\sigma_3 & 0 \\ 0 & -i\sigma_3 & 0 & 0 \\ i\sigma_3 & 0 & 0 & 0 \end{pmatrix}, \tag{27}$$

$$\mathfrak{e}_3 = \begin{pmatrix} 0 & 0 & 0 & i\sigma_2 \\ 0 & 0 & i\sigma_2 & 0 \\ 0 & -i\sigma_2 & 0 & 0 \\ -i\sigma_2 & 0 & 0 & 0 \end{pmatrix}, \tilde{\mathfrak{e}}_3 = \begin{pmatrix} 0 & 0 & 0 & -i\sigma_1 \\ 0 & 0 & i\sigma_1 & 0 \\ 0 & -i\sigma_1 & 0 & 0 \\ i\sigma_1 & 0 & 0 & 0 \end{pmatrix}. \tag{28}$$

Matrix representation of $\mathcal{A}_{\mathrm{SM}}$ on its ideal $\bigwedge^\bullet \chi^\dagger \mathfrak{p}$

We define the elements

$$\begin{cases} \mathfrak{e} := & \mathfrak{e}_1\mathfrak{e}_2\mathfrak{e}_3, \\ \tilde{\mathfrak{e}} = & \tilde{\mathfrak{e}}_1\tilde{\mathfrak{e}}_2\tilde{\mathfrak{e}}_3. \end{cases} \tag{29}$$

Then, $\mathfrak{e}^2 = -1$, $\tilde{\mathfrak{e}}^2 = -1$, $\mathfrak{e}\tilde{\mathfrak{e}} = -\tilde{\mathfrak{e}}\mathfrak{e}$, and $(\mathfrak{e}\tilde{\mathfrak{e}})^2 = -1$.

The matrix representations of the elements $\mathfrak{e}$, $\tilde{\mathfrak{e}}$, and $\mathfrak{e}\tilde{\mathfrak{e}}$ is

$$\mathfrak{e} = \begin{pmatrix} 0_4 & 1_4 \\ -1_4 & 0_4 \end{pmatrix}, \tilde{\mathfrak{e}} = i\begin{pmatrix} 0_4 & 1_4 \\ 1_4 & 0_4 \end{pmatrix}, \mathfrak{e}\tilde{\mathfrak{e}} = i\begin{pmatrix} 1_4 & 0_4 \\ 0_4 & -1_4 \end{pmatrix}. \tag{30}$$

The ideal decomposition of $\mathcal{A}_{\mathrm{SM}}$

It is helpful sometimes to use multiindices $K \subset \{1, 2, 3\}$. This allows us to write immediately a matrix representation of the algebra $\mathcal{A}_{\mathrm{SM}}$. We can represent the spinors from $\bigwedge^{\bullet} \chi^\dagger \mathfrak{q}$ as vectors

$$\Psi \mathfrak{q} = \sum_{K \subset \{1,2,3\}} \Psi^K \mathfrak{q}^\dagger{}_K \mathfrak{q}, \tag{31}$$

where $\Psi^K \in \mathbb{C}$. Similarly, their duals can be expressed in the following vector form

$$\mathfrak{q}^\dagger \Psi^\dagger = \sum_{K \subset \{1,2,3\}} \Psi^\dagger_K \mathfrak{q}^\dagger \mathfrak{q}_K, \tag{32}$$

where $\Psi^\dagger_K \in \mathbb{C}$.

Any element a of $\mathcal{A}_{\mathrm{SM}}$ can be written uniquely as a linear combination of the form

$$a = \sum_{K_1, K_2 \subset \{1,2,3\}} a^{K_1}{}_{K_2}\ \mathfrak{q}^\dagger{}_{K_1}\ \mathfrak{p}\ \mathfrak{q}_{K_2}. \tag{33}$$

The ideal decomposition of $\mathcal{A}_{\mathsf{SM}}$

Therefore, the Witt decomposition $\mathcal{A}^1_{\mathsf{SM}} = \chi^\dagger \oplus \chi$ gives a natural decomposition of $\mathcal{A}_{\mathsf{SM}}$ as a direct sum of left ideals

$$\mathcal{A}_{\mathsf{SM}} = \bigoplus_{k=0}^{3} \left(\bigwedge{}^{\bullet}\chi^\dagger\right) \mathfrak{p} \bigwedge{}^{k}\chi, \tag{34}$$

which means that $\mathcal{A}_{\mathsf{SM}}$ decomposes as sum of spinors with internal degrees of freedom in $\bigwedge^k\chi$, similar to leptons and quarks.

The Dirac algebra

The representation of the Dirac algebra on a minimal left ideal is decomposed by the projectors $\frac{1}{2}(1 \pm i\mathfrak{e}\tilde{\mathfrak{e}})$ into two irreducible representations.

In addition, each of the resulting four-dimensional subspaces has to be split into complex two-dimensional spaces corresponding to chirality. So we need the representation of the chirality operator, which we take to be

$$\Gamma^5 := -i\mathfrak{e}_1\tilde{\mathfrak{e}}_1 = \begin{pmatrix} 1_2 & 0 & 0 & 0 \\ 0 & -1_2 & 0 & 0 \\ 0 & 0 & -1_2 & 0 \\ 0 & 0 & 0 & 1_2 \end{pmatrix}. \tag{35}$$

The Dirac algebra

Let us recall the chiral (Weyl) representation,

$$\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} -1_2 & 0 \\ 0 & 1_2 \end{pmatrix} \tag{36}$$

and define a modified version of it, $\widetilde{\gamma}^0 = \gamma^0$, $\widetilde{\gamma}^j = -\gamma^j$, $\widetilde{\gamma}^5 = -\gamma^5$.

Then the Dirac representation on the eight-dimensional space $\mathcal{A}_{\mathsf{SM}}\mathfrak{p}$ is the direct sum of the two chiral representations,

$$\Gamma^\mu = \begin{pmatrix} \widetilde{\gamma}^\mu & 0 \\ 0 & \gamma^\mu \end{pmatrix}. \tag{37}$$

This choice will turn out to be convenient when talking about the weak interaction.

Weak symmetry in the Standard Model Algebra

Weak symmetry in the Standard Model Algebra

Now we look for the representations of $\mathsf{SU}(2)_\mathsf{L}$, taking into account the chirality of each space. Consider the elements

$$\omega_\mathsf{u} = \begin{pmatrix} 0 & 1_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \omega_\mathsf{d} = \begin{pmatrix} 0 & 0 & -1_2 & 0 \\ 0 & 0 & 0 & -1_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{38}$$

$$\omega_\circ = \begin{pmatrix} \sigma_+ & 0 & 0 & 0 \\ 0 & -\sigma_+ & 0 & 0 \\ 0 & 0 & -\sigma_+ & 0 \\ 0 & 0 & 0 & \sigma_+ \end{pmatrix}. \tag{39}$$

Weak symmetry in the Standard Model Algebra

We define the null complex vector spaces $\mathscr{N}$ and $\mathscr{N}^\dagger$ as the spaces spanned by null vectors from (41), by

$$\begin{cases} \mathscr{N} := \mathsf{span}_{\mathbb{C}}\left(\omega_{\mathsf{u}}, \omega_{\mathsf{d}}, \omega_{\circ}\right), \\ \mathscr{N}^\dagger := \mathsf{span}_{\mathbb{C}}\left(\omega^\dagger_{\mathsf{u}}, \omega^\dagger_{\mathsf{d}}, \omega^\dagger_{\circ}\right). \end{cases} \tag{40}$$

The elements

$$\left(\omega_{\mathsf{u}}, \omega_{\mathsf{d}}, \omega_{\circ}, \omega^\dagger_{\mathsf{u}}, \omega^\dagger_{\mathsf{d}}, \omega^\dagger_{\circ}\right) \tag{41}$$

form a *Witt basis* of the space $\mathscr{N}^\dagger \oplus \mathscr{N}$, satisfying the *anticommutation relations*

$$\begin{aligned} \{\omega_j, \omega_k\} &= 0, \\ \{\omega^\dagger_j, \omega^\dagger_k\} &= 0, \\ \{\omega_j, \omega^\dagger_k\} &= \delta_{jk} \end{aligned} \tag{42}$$

for $j, k \in \{\mathsf{u}, \mathsf{d}, \circ\}$.

Weak symmetry in the Standard Model Algebra

We define the orthonormal basis

$$\begin{cases} \mathfrak{u}_j = \omega_j + \omega^\dagger{}_j \\ \mathfrak{u}'_j = i\left(\omega^\dagger{}_j - \omega_j\right), \end{cases} \tag{43}$$

where $j \in \{\mathrm{u}, \mathrm{d}, \circ\}$.

Then, $\mathfrak{u}_j^2 = 1$, $\mathfrak{u}_j'^2 = 1$, and

$$\begin{cases} \omega_j = \frac{1}{2}\left(\mathfrak{u}_j + i\mathfrak{u}'_j\right) \\ \omega^\dagger{}_j = \frac{1}{2}\left(\mathfrak{u}_j - i\mathfrak{u}'_j\right). \end{cases} \tag{44}$$

Weak symmetry in the Standard Model Algebra

The matrix form of $\mathfrak{u}_j$ and $\mathfrak{u}'_j$ is

$$\mathfrak{u}_\mathsf{u} = \begin{pmatrix} 0 & 1_2 & 0 & 0 \\ 1_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1_2 \\ 0 & 0 & -1_2 & 0 \end{pmatrix}, \mathfrak{u}'_\mathsf{u} = \begin{pmatrix} 0 & -i1_2 & 0 & 0 \\ i1_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & i1_2 \\ 0 & 0 & -i1_2 & 0 \end{pmatrix}, \quad (45)$$

$$\mathfrak{u}_\mathsf{d} = \begin{pmatrix} 0 & 0 & -1_2 & 0 \\ 0 & 0 & 0 & -1_2 \\ -1_2 & 0 & 0 & 0 \\ 0 & -1_2 & 0 & 0 \end{pmatrix}, \mathfrak{u}'_\mathsf{d} = \begin{pmatrix} 0 & 0 & i1_2 & 0 \\ 0 & 0 & 0 & i1_2 \\ -i1_2 & 0 & 0 & 0 \\ 0 & -i1_2 & 0 & 0 \end{pmatrix}, \quad (46)$$

$$\mathfrak{u}_\circ = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & -\sigma_1 & 0 & 0 \\ 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 0 & \sigma_1 \end{pmatrix}, \mathfrak{u}'_\circ = \begin{pmatrix} \sigma_2 & 0 & 0 & 0 \\ 0 & -\sigma_2 & 0 & 0 \\ 0 & 0 & -\sigma_2 & 0 \\ 0 & 0 & 0 & \sigma_2 \end{pmatrix}. \quad (47)$$

Weak symmetry in the Standard Model Algebra

None of the elements $\mathfrak{u}_j, \mathfrak{u}'_j, \omega^\dagger{}_j, \omega_j$ are linear combinations of the elements $(\mathfrak{q}^\dagger{}_1, \mathfrak{q}^\dagger{}_2, \mathfrak{q}^\dagger{}_3, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3)$. Then,

$$\mathcal{N}^\dagger, \mathcal{N} \neq \chi^\dagger, \chi.$$

The elements

$$\begin{cases} \omega := & \omega_{\mathsf{u}}\omega_{\mathsf{d}}\omega_{\circ}, \\ \omega^\dagger = & \omega^\dagger{}_{\circ}\omega^\dagger{}_{\mathsf{d}}\omega^\dagger{}_{\mathsf{u}} \end{cases} \tag{48}$$

are *nilpotent*, $\omega^2 = 0$ and $\omega^{\dagger 2} = 0$.

The nilpotents ω and $\omega^\dagger$ have the following matrix form

$$\omega = \begin{pmatrix} 0 & 0 & 0 & -\sigma_+ \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \omega^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\sigma_- & 0 & 0 & 0 \end{pmatrix}. \tag{49}$$

Weak symmetry in the Standard Model Algebra

From them we can construct the *idempotents* $\omega^\dagger \omega$ and $\omega\omega^\dagger$,

$$\omega\omega^\dagger = \mathfrak{p} = \begin{pmatrix} \sigma_3^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \omega^\dagger \omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_3^- \end{pmatrix}. \tag{50}$$

Then,

$$\mathcal{A}_{\mathsf{SM}}\mathfrak{p} = \bigwedge{}^{\bullet}\mathcal{N}^\dagger \mathfrak{p}. \tag{51}$$

The vectors ω_j and $\omega_j^\dagger$ act as ladder operators on this ideal, similar to (18):

$$\begin{cases} \omega_j^\dagger(a\mathfrak{p}) = (\omega_j^\dagger \wedge a)\mathfrak{p}, \\ \omega_j(a\mathfrak{p}) = (i_{\omega_j} a)\mathfrak{p}, \end{cases} \tag{52}$$

where $a \in \bigwedge{}^{\bullet}\mathcal{N}^\dagger$, and i_{ω_j} is the interior product defined by the Hermitian inner product $h_{\mathcal{N}}$. This definition is consistent with the anticommutation relations (42).

Weak symmetry in the Standard Model Algebra

From the relations (52) it follows that the matrix form (38) corresponds to the basis

$$\left(1\ \mathfrak{p}, {\omega^\dagger}_\circ\ \mathfrak{p}, {\omega^\dagger}_\mathsf{u}\ \mathfrak{p}, {\omega^\dagger}_\mathsf{u}{\omega^\dagger}_\circ\ \mathfrak{p}, {\omega^\dagger}_\mathsf{d}\ \mathfrak{p}, {\omega^\dagger}_\mathsf{d}{\omega^\dagger}_\circ\ \mathfrak{p}, {\omega^\dagger}_\mathsf{d}{\omega^\dagger}_\mathsf{u}\ \mathfrak{p}, {\omega^\dagger}_\mathsf{d}{\omega^\dagger}_\mathsf{u}{\omega^\dagger}_\circ\ \mathfrak{p}\right). \tag{53}$$

At the same time, the matrices (38) are expressed in the basis (19). Hence,

$$\left\{\begin{array}{rcr} {\omega^\dagger}_\circ\ \mathfrak{p} & = & {\mathfrak{q}^\dagger}_{23}\ \mathfrak{p} \\ {\omega^\dagger}_\mathsf{u}\ \mathfrak{p} & = & {\mathfrak{q}^\dagger}_{31}\ \mathfrak{p} \\ {\omega^\dagger}_\mathsf{u}{\omega^\dagger}_\circ\ \mathfrak{p} & = & {\mathfrak{q}^\dagger}_{12}\ \mathfrak{p} \\ {\omega^\dagger}_\mathsf{d}\ \mathfrak{p} & = & {\mathfrak{q}^\dagger}_{321}\ \mathfrak{p} \\ {\omega^\dagger}_\mathsf{d}{\omega^\dagger}_\circ\ \mathfrak{p} & = & {\mathfrak{q}^\dagger}_{1}\ \mathfrak{p} \\ {\omega^\dagger}_\mathsf{d}{\omega^\dagger}_\mathsf{u}\ \mathfrak{p} & = & {\mathfrak{q}^\dagger}_{2}\ \mathfrak{p} \\ {\omega^\dagger}_\mathsf{d}{\omega^\dagger}_\mathsf{u}{\omega^\dagger}_\circ\ \mathfrak{p} & = & {\mathfrak{q}^\dagger}_{3}\ \mathfrak{p} \end{array}\right. \tag{54}$$

Although the identities (54) are between elements of the same ideal $\bigwedge^\bullet \mathscr{N}^\dagger \mathfrak{p} = \bigwedge^\bullet \chi^\dagger \mathfrak{p}$, the spaces $\bigwedge^\bullet \mathscr{N}^\dagger$ and $\bigwedge^\bullet \chi^\dagger$ are different.

The weak symmetry – Spinorial generators

The weak symmetry – Spinorial generators

Let $\mathbb{W}_{0\mathsf{R}} := \mathrm{span}_{\mathbb{C}}\left(\mathfrak{p}, \omega^{\dagger}{}_{\circ}\mathfrak{p}\right)$ be the vector subspace of the ideal $\mathcal{A}_{\mathsf{SM}}\mathfrak{p}$ spanned by the null vectors $\mathfrak{p}$ and $\omega^{\dagger}{}_{\circ}\mathfrak{p}$. In the following, it will correspond to the *up* particle singlet space of the weak symmetry. The elements of the basis (53) split the ideal $\mathcal{A}_{\mathsf{SM}}\mathfrak{p}$ into subspaces which correspond to the singlets and doublets of the weak symmetry:

$$\begin{cases} \text{Right-handed up singlet space: } \mathbb{W}_{0\mathsf{R}} := 1\,\mathrm{span}_{\mathbb{C}}\left(\mathfrak{p}, \omega^{\dagger}{}_{\circ}\mathfrak{p}\right), \\ \text{Left-handed up doublet space: } \mathbb{W}_{0\mathsf{L}} := \omega^{\dagger}{}_{\mathsf{u}}\mathbb{W}_{0\mathsf{R}}, \\ \text{Right-handed down singlet space: } \mathbb{W}_{1\mathsf{R}} := \omega^{\dagger}{}_{\mathsf{u}}\omega^{\dagger}{}_{\mathsf{d}}\mathbb{W}_{0\mathsf{R}}, \\ \text{Left-handed down doublet space: } \mathbb{W}_{1\mathsf{L}} := \omega^{\dagger}{}_{\mathsf{d}}\mathbb{W}_{0\mathsf{R}}. \end{cases} \tag{55}$$

The Clifford algebra $\mathcal{A}_{\mathsf{SM}}$ contains a spin representation of the weak group $\mathsf{SU}(2)_{\mathsf{L}}$, which is a double cover of the representation normally used.

The weak symmetry – Spinorial generators

We choose the following set of generator bivectors for the group $SU(2)_L$:

$$\begin{cases} \widetilde{T}_1 := u_u u'_d - u'_u u_d \\ \widetilde{T}_2 := u_u u_d + u'_u u'_d \\ \widetilde{T}_3 := u_u u'_u - u_d u'_d \end{cases} \tag{56}$$

They have the following matrix form in the basis (53):

$$\widetilde{T}_1 = 2i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1_2 & 0 \\ 0 & 1_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \widetilde{T}_2 = 2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1_2 & 0 \\ 0 & 1_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{57}$$

$$\widetilde{T}_3 = 2i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1_2 & 0 & 0 \\ 0 & 0 & 1_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{58}$$

The weak symmetry – Spinorial generators

The proof that the bivectors in equation (56) are spinorial generators of the $\mathrm{SU}(2)_{\mathrm{L}}$ group is given in (Stoica 2018),

where in addition it is shown that $\sin^2\theta_W = 0.25$, which seems more encouraging that that of 0.375 predicted by the $\mathrm{SU}(5)$, $\mathrm{Spin}(10)$, and other GUTs.

Depending on the utilized scheme, the experimental values for $\sin^2\theta_W$, range between ~ 0.223 and ~ 0.24 (Erler and Freitas 2015; Mohr and Newe 2016).

The electrocolor symmetry

The electrocolor symmetry

A set of generator bivectors for the group $\mathrm{SU}(3)_c$ can be chosen to correspond to the *Gell-Mann matrices*,

$$\begin{aligned}
&\widetilde{\lambda}_1 = \mathfrak{e}_1\tilde{\mathfrak{e}}_2 - \tilde{\mathfrak{e}}_1\mathfrak{e}_2, \quad \widetilde{\lambda}_2 = \mathfrak{e}_1\mathfrak{e}_2 + \tilde{\mathfrak{e}}_1\tilde{\mathfrak{e}}_2, \quad \widetilde{\lambda}_3 = \mathfrak{e}_1\tilde{\mathfrak{e}}_1 - \mathfrak{e}_2\tilde{\mathfrak{e}}_2, \\
&\widetilde{\lambda}_4 = \mathfrak{e}_1\tilde{\mathfrak{e}}_3 - \tilde{\mathfrak{e}}_1\mathfrak{e}_3, \quad \widetilde{\lambda}_5 = \mathfrak{e}_1\mathfrak{e}_3 + \tilde{\mathfrak{e}}_1\tilde{\mathfrak{e}}_3, \\
&\widetilde{\lambda}_6 = \mathfrak{e}_2\tilde{\mathfrak{e}}_3 - \tilde{\mathfrak{e}}_2\mathfrak{e}_3, \quad \widetilde{\lambda}_7 = \mathfrak{e}_2\mathfrak{e}_3 + \tilde{\mathfrak{e}}_2\tilde{\mathfrak{e}}_3, \quad \widetilde{\lambda}_8 = \frac{1}{\sqrt{3}}(\mathfrak{e}_1\tilde{\mathfrak{e}}_1 + \mathfrak{e}_2\tilde{\mathfrak{e}}_2 - 2\mathfrak{e}_3\tilde{\mathfrak{e}}_3).
\end{aligned} \tag{59}$$

The generator of $\mathrm{U}(1)_{\mathrm{em}}$ is the identity of $\mathrm{End}_{\mathbb{C}}(\chi)$,

$$Q = \mathfrak{e}_1\tilde{\mathfrak{e}}_1 + \mathfrak{e}_2\tilde{\mathfrak{e}}_2 + \mathfrak{e}_3\tilde{\mathfrak{e}}_3. \tag{60}$$

It is immediate to see that $\widetilde{\lambda}_j^\dagger = -\widetilde{\lambda}_j$ for all values of j.

The electrocolor symmetry

The standard Gell-Mann matrices are defined by $\lambda_j = i\widetilde{\lambda}_j$. Then,

$$e^{-i\varphi\lambda_j} a = e^{\frac{\varphi}{2}\widetilde{\lambda}j} a e^{-\frac{\varphi}{2}\widetilde{\lambda}j}, \tag{61}$$

for the $\mathrm{SU}(3)_c$ representation **3**.

Since the action of an element $g \in \mathrm{Spin}(\chi^\dagger \oplus \chi)$ on an element $\omega \in \mathcal{A}_{\mathrm{SM}}$ is given by $\omega \mapsto g\omega g^{-1}$, the action of $\mathrm{SU}(3)_c$ and $\mathrm{U}(1)_{\mathrm{em}}$ on χ extends to the exterior algebra $\bigwedge^\bullet \chi$, in a way compatible with the exterior product. Hence, these spinorial generators give the right representations for the color and electric charge.

The symmetry generated by (60) transforms not only $\mathfrak{p} \bigwedge^k \chi$, but also $\omega_{\mathrm{d}}^\dagger \mathfrak{p}$. From $\omega_{\mathrm{d}}^\dagger \mathfrak{p} = \mathfrak{q}^\dagger \mathfrak{p}$ it follows that the electric charge of $\mathfrak{q}^\dagger \mathfrak{p}$ is -1. This accounts for the fact that each minimal left ideal contains two different particles, with different electric charges.

Leptons and quarks

Leptons and quarks

From

$$\mathcal{A}_{\mathsf{SM}} = \bigoplus_{k=0}^{3} \mathcal{A}_{\mathsf{SM}}\mathfrak{p} \bigwedge{}^{k}\chi. \tag{62}$$

and

$$\mathcal{A}_{\mathsf{SM}}\mathfrak{p} = \mathbb{W}_0 \oplus \mathbb{W}_1 = \mathbb{W}_{0\mathsf{R}} \oplus \mathbb{W}_{0\mathsf{L}} \oplus \mathbb{W}_{1\mathsf{L}} \oplus \mathbb{W}_{1\mathsf{R}}. \tag{63}$$

it follows that

$$\mathcal{A}_{\mathsf{SM}} = \bigoplus_{k=0}^{3} (\mathbb{W}_0 \oplus \mathbb{W}_1) \bigwedge{}^{k}\chi, \tag{64}$$

and in terms of the chiral spaces,

$$\mathcal{A}_{\mathsf{SM}} = \bigoplus_{k=0}^{3} (\mathbb{W}_{0\mathsf{R}} \oplus \mathbb{W}_{0\mathsf{L}} \oplus \mathbb{W}_{1\mathsf{L}} \oplus \mathbb{W}_{1\mathsf{R}}) \bigwedge{}^{k}\chi. \tag{65}$$

Leptons and quarks

We centralize all these remarks, and use as classifiers the elements of the form $\mathfrak{p}\mathfrak{q}_K$ and $\mathfrak{q}^\dagger\mathfrak{p}\mathfrak{q}_K$.

Then, the data in this table

Particle		$\mathbf{e}^-$	$\overline{\mathbf{u}}$	$\mathbf{d}$	$\overline{\nu}$	ν	$\overline{\mathbf{d}}$	$\mathbf{u}$	$\mathbf{e}^+$
Electric charge		-1	$-\frac{2}{3}$	$-\frac{1}{3}$	0	0	$+\frac{1}{3}$	$+\frac{2}{3}$	$+1$
Weak isospin	**L**	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$+\frac{1}{2}$	0	$+\frac{1}{2}$	0
	R	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$+\frac{1}{2}$	0	$+\frac{1}{2}$
Hypercharge	**L**	-1	$-\frac{4}{3}$	$+\frac{1}{3}$	0	-1	$+\frac{2}{3}$	$+\frac{1}{3}$	$+2$
	R	-2	$-\frac{1}{3}$	$-\frac{2}{3}$	1	0	$-\frac{1}{3}$	$+\frac{4}{3}$	$+1$

Leptons and quarks

can be classified as in the following table:

Particle		ν	$\overline{\mathbf{d}}$	$\mathbf{u}$	$\mathbf{e}^+$
Spinor space		$\mathbb{W}_0$	$\mathbb{W}_0\mathfrak{q}_j$	$\mathbb{W}_0\mathfrak{q}_{jk}$	$\mathbb{W}_0\mathfrak{q}_{123}$
Classifier		$\mathfrak{p}$	$\mathfrak{p}\mathfrak{q}_j$	$\mathfrak{p}\mathfrak{q}_{jk}$	$\mathfrak{p}\mathfrak{q}_{123}$
Electric charge		0	$+\frac{1}{3}$	$+\frac{2}{3}$	$+1$
Chiral space	**L**	$\omega_{\mathrm{u}}^{\dagger}\mathbb{W}_0$	$\mathbb{W}_0\mathfrak{q}_j$	$\omega_{\mathrm{u}}^{\dagger}\mathbb{W}_0\mathfrak{q}_{jk}$	$\mathbb{W}_0\mathfrak{q}_{123}$
	R	$\mathbb{W}_0$	$\omega_{\mathrm{u}}^{\dagger}\mathbb{W}_0\mathfrak{q}_j$	$\mathbb{W}_0\mathfrak{q}_{jk}$	$\omega_{\mathrm{u}}^{\dagger}\mathbb{W}_0\mathfrak{q}_{123}$
Weak isospin	**L**	$+\frac{1}{2}$	0	$+\frac{1}{2}$	0
	R	0	$+\frac{1}{2}$	0	$+\frac{1}{2}$
Hypercharge	**L**	-1	$+\frac{2}{3}$	$+\frac{1}{3}$	$+2$
	R	0	$-\frac{1}{3}$	$+\frac{4}{3}$	$+1$
Particle		$\mathbf{e}^-$	$\overline{\mathbf{u}}$	$\mathbf{d}$	$\overline{\nu}$
Spinor space		$\mathbb{W}_1$	$\mathbb{W}_1\mathfrak{q}_j$	$\mathbb{W}_1\mathfrak{q}_{jk}$	$\mathbb{W}_1\mathfrak{q}_{123}$
Classifier		$\mathfrak{q}^{\dagger}\mathfrak{p}$	$\mathfrak{q}^{\dagger}\mathfrak{p}\mathfrak{q}_j$	$\mathfrak{q}^{\dagger}\mathfrak{p}\mathfrak{q}_{jk}$	$\mathfrak{q}^{\dagger}\mathfrak{p}\mathfrak{q}_{123}$
Electric charge		-1	$-\frac{2}{3}$	$-\frac{1}{3}$	0
Chiral space	**L**	$\omega_{\mathrm{d}}^{\dagger}\mathbb{W}_0$	$\omega_{\mathrm{u}}^{\dagger}\omega_{\mathrm{d}}^{\dagger}\mathbb{W}_0\mathfrak{q}_j$	$\omega_{\mathrm{d}}^{\dagger}\mathbb{W}_0\mathfrak{q}_{jk}$	$\omega_{\mathrm{u}}^{\dagger}\omega_{\mathrm{d}}^{\dagger}\mathbb{W}_0\mathfrak{q}_{123}$
	R	$\omega_{\mathrm{u}}^{\dagger}\omega_{\mathrm{d}}^{\dagger}\mathbb{W}_0$	$\omega_{\mathrm{d}}^{\dagger}\mathbb{W}_0\mathfrak{q}_j$	$\omega_{\mathrm{u}}^{\dagger}\omega_{\mathrm{d}}^{\dagger}\mathbb{W}_0\mathfrak{q}_{jk}$	$\omega_{\mathrm{d}}^{\dagger}\mathbb{W}_0\mathfrak{q}_{123}$
Weak isospin	**L**	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0
	R	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
Hypercharge	**L**	-1	$-\frac{4}{3}$	$+\frac{1}{3}$	0
	R	-2	$-\frac{1}{3}$	$-\frac{2}{3}$	1

Leptons and quarks

and have the matrix form

		$\mathbf{1}_c$	$\mathbf{3}_c$			$\overline{\mathbf{1}}_c$	$\overline{\mathbf{3}}_c$		
Dirac, Lorentz	$\mathbf{1}_w$	ν_{R1}	u^r_{R1}	u^y_{R1}	u^b_{R1}	$\bar{e}_{L1}$	$\bar{d}^r_{L1}$	$\bar{d}^y_{L1}$	$\bar{d}^b_{L1}$
		ν_{R2}	u^r_{R2}	u^y_{R2}	u^b_{R2}	$\bar{e}_{L2}$	$\bar{d}^r_{L2}$	$\bar{d}^y_{L2}$	$\bar{d}^b_{L2}$
	$\mathbf{2}_w$	ν_{L1}	u^r_{L1}	u^y_{L1}	u^b_{L1}	$\bar{e}_{R1}$	$\bar{d}^r_{R1}$	$\bar{d}^y_{R1}$	$\bar{d}^b_{R1}$
		ν_{L2}	u^r_{L2}	u^y_{L2}	u^b_{L2}	$\bar{e}_{R2}$	$\bar{d}^r_{R2}$	$\bar{d}^y_{R2}$	$\bar{d}^b_{R2}$
Dirac, Lorentz		e_{L1}	d^r_{L1}	d^y_{L1}	d^b_{L1}	$\bar{\nu}_{R1}$	$\bar{u}^r_{R1}$	$\bar{u}^y_{R1}$	$\bar{u}^b_{R1}$
		e_{L2}	d^r_{L2}	d^y_{L2}	d^b_{L2}	$\bar{\nu}_{R2}$	$\bar{u}^r_{R2}$	$\bar{u}^y_{R2}$	$\bar{u}^b_{R2}$
	$\mathbf{1}_w$	e_{R1}	d^r_{R1}	d^y_{R1}	d^b_{R1}	$\bar{\nu}_{L1}$	$\bar{u}^r_{L1}$	$\bar{u}^y_{L1}$	$\bar{u}^b_{L1}$
		e_{R2}	d^r_{R2}	d^y_{R2}	d^b_{R2}	$\bar{\nu}_{L2}$	$\bar{u}^r_{L2}$	$\bar{u}^y_{L2}$	$\bar{u}^b_{L2}$

All symmetries

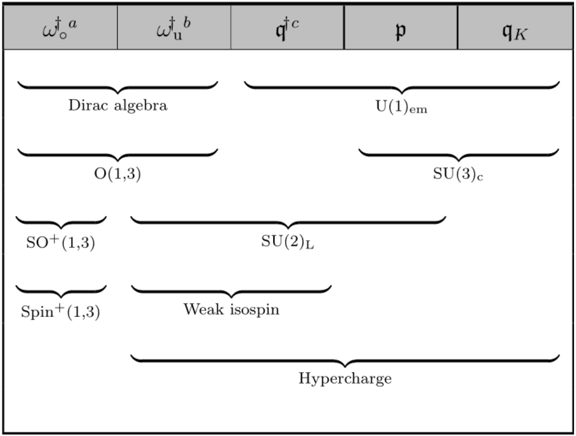
All symmetries

Any element of the $\mathcal{A}_{\mathrm{SM}}$ is a linear combination of elements of the form

$$\omega^{\dagger}{}_{\circ}{}^{a}\,\omega^{\dagger}{}_{\mathbf{u}}{}^{b}\,\mathfrak{q}^{\dagger\,c}\,\mathfrak{p}\,\mathfrak{q}_{K}, \tag{66}$$

where $K \subset \{1,2,3\}$ is a multiindex, $a, b, c \in \{0,1\}$, and by convention, $(\omega^{\dagger}{}_{\mathbf{u}})^{0} = (\omega^{\dagger}{}_{\circ})^{0} = (\mathfrak{q}^{\dagger})^{0} = 1$.

Ranges of various actions on the Standard Model Algebra:



We see the overlap between improper Lorentz transformations and the weak symmetry group.

Thank you!

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Appendix 1: Relations with other models

Relations with other models

The SMA model shares common features with previously known models. Particles of two distinct flavors were previously combined into 8-spinor ideals, in a unified spin gauge theory of gravity and electroweak interactions based on $C\ell_{1,6} \cong C\ell_{1,3} \otimes C\ell_{0,3}$ (Chisholm and Farwell 1996), and in (Trayling 1999; Trayling and Baylis 2001; Trayling and Baylis 2004) based on $C\ell_7 \cong C\ell_3 \otimes C\ell_4$, where there are three space dimensions, the time is a scalar, the four extra dimensions related to the Higgs boson, the predicted Weinberg angle is given by $\sin^2\theta_W = 0.375$, and remarkably, the full symmetries of the SM arise from the condition to preserve the current and to leave right-handed neutrino sterile.

Relations with other models

Among the main differences, the SMA model uses different structures, leading to the algebra $\mathbb{C}\ell(\chi^\dagger \oplus \chi) \cong \mathbb{C}\ell_6$, includes the Dirac algebra $C\ell_{1,3} \otimes \mathbb{C}$, and $\sin^2 \theta_W = 0.25$. In the $C\ell_{1,6}$ and $C\ell_7$ models the ideals are obtained using primitive idempotents. The SMA model uses a decomposition into left ideals $\mathbb{C}\ell_6 \mathfrak{q}\mathfrak{q}^\dagger \mathfrak{q}_K$, where $\mathbb{K} \subseteq \{1,2,3\}$ (notations from §3), based on the Witt decomposition $\chi^\dagger \oplus \chi$ and the exterior algebra $\bigwedge^k \chi$ contained within the minimal right ideal $\mathfrak{q}\mathfrak{q}^\dagger \mathbb{C}\ell_6 = \mathfrak{q}\mathfrak{q}^\dagger \bigwedge^k \chi$. This allows the colors and charges to be emphasized, and the minimal left ideals of the same charge and different colors to be coupled into a larger ideal.

Relations with other models

I arrived at the symmetries $\mathsf{SU}(3)_c$ and $\mathsf{U}(1)_{em}$ and the generators (59) and (60) starting from the standard ideal decomposition of Clifford algebras $\mathbb{C}\ell_{2r}$ (Chevalley 1997; Crumeyrolle 1990), the representation of $\mathsf{U}(N)$ and $\mathsf{SU}(N)$ on $C\ell_{2N}$ as the subgroup of $\mathsf{Spin}(2N)$ preserving a Hermitian inner product, given in (Doran et al. 1993), and by the standard construction of the Hermitian exterior algebra (**ROWells2007ComplexManifolds**), resulting in the correct $\mathbf{1}_c, \mathbf{3}_c, \overline{\mathbf{1}}_c$, ad $\overline{\mathbf{3}}_c$ representations. A proof that the unitary spin transformations preserving a Witt decomposition in $\mathbb{C}\ell_6$ give the $\mathsf{SU}(3)_c$ and $\mathsf{U}(1)_{em}$ symmetries, along with a set of generators constructed from the $\mathfrak{q}_j$ and $\mathfrak{q}_j^\dagger$ ladder operators but equivalent to (59), was given in (Furey 2015). Based on the algebra $C\ell_7$, in (Trayling and Baylis 2004) were proposed generators of $\mathsf{SU}(3)_c$ which are equivalent to (59) due to the isomorphisms $C\ell_7 \cong \mathbf{M}_{\mathbb{C}}(8) \cong \mathbb{C}\ell_6$.

Relations with other models

In a model based on octonions, Furey (Furey 2015; Furey 2016) uses the Witt decomposition for $\mathbb{C}\ell_6$ to represent colors and charges of up- and down-type particles by $\mathfrak{q}^\dagger{}_K\mathfrak{q}\mathfrak{q}^\dagger$ and $\mathfrak{q}_K\mathfrak{q}^\dagger\mathfrak{q}$, on the minimal left ideals $\mathbb{C}\ell_6\mathfrak{q}\mathfrak{q}^\dagger$ and $\mathbb{C}\ell_6\mathfrak{q}^\dagger\mathfrak{q}$. They are united into a single irreducible representation of $\mathbb{C}\ell_6 \otimes_{\mathbb{C}} \mathbb{C}\ell_2$ obtained by using the octonion algebra. To represent the complete particles, with spin and chirality, Furey proposes including the quaternion algebra, resulting in a representation of leptons and quarks as spinors of an algebra isomorphic to $\mathbb{C}\ell_{12}$. By contrast, in the SMA model, everything is contained in the ideals of $\mathbb{C}\ell_6$ classified by the elements $\mathfrak{q}\mathfrak{q}^\dagger\mathfrak{q}_K$. Despite these differences, the $\mathrm{SU}(3)_c$ and $\mathrm{U}(1)_{em}$ symmetries in the SMA are identical to those obtained previously by Furey (Furey 2016) as the unitary spin transformations preserving the Witt decomposition of $\mathbb{C}\ell_6$, improving by this previous results based on octonions and Clifford algebras (Günaydin and Gürsey 1974; Barducci et al. 1977; Casalbuoni and Gatto 1979).

Appendix 2: The Weinberg Angle

The Weinberg Angle i

In the following I discuss the electroweak symmetry breaking from geometric point of view. I will review first the geometry of the standard electroweak symmetry breaking in a way similar to (Derdzinski 1992, Ch. 6).

Then, I will calculate the Weinberg angle as seems to be predicted by the Standard Model Algebra.

The exchange bosons of the electroweak force are connections in the gauge bundle having as fiber the two-dimensional Hermitian vector space $(\mathbb{W}_w, h_{\mathrm{w}})$, where $\mathbb{W}_w := \mathrm{span}_{\mathbb{C}}\left(\omega^{\dagger}{}_{\mathtt{u}}, \omega^{\dagger}{}_{\mathtt{d}}\right)$.

Consequently, the internal components of the exchange bosons of the electroweak force are elements of the unitary Lie algebra $\mathfrak{u}(2)_{\mathrm{ew}} \cong \mathfrak{u}\left(\mathbb{W}_w\right)$, that is, Hermitian forms.

The Weinberg Angle ii

The unitary Lie algebra $\mathfrak{u}(2)_{\mathrm{ew}}$, regarded as a vector space, has four real dimensions. After the symmetry breaking, they correspond to the photon γ, and the weak force bosons $W^{\pm}$ and Z^0.

Following (Derdzinski 1992), the decomposition of the Lie algebra $\mathfrak{u}(2)_{\mathrm{ew}}$ into subspaces where each of these bosons live is

$$\mathfrak{u}\left(\mathbb{W}_w\right) = \gamma\left(\mathbb{W}_w\right) \oplus W\left(\mathbb{W}_w\right) \oplus Z\left(\mathbb{W}_w\right). \tag{67}$$

Hence, $\gamma \in \gamma\left(\mathbb{W}_w\right)$, $W^{\pm} \in W\left(\mathbb{W}_w\right)$, and $Z^0 \in Z\left(\mathbb{W}_w\right)$.

The decomposition (67) is not unique, but is uniquely determined by the *Higgs field* ϕ and the *Weinberg electroweak mixing angle* θ_W.

The Weinberg Angle iii

In fact, what we need is a special complex line in the space $\mathbb{W}_w$, which is determined by ϕ, and an Ad-invariant inner product on $\mathfrak{u}(\mathbb{W}_w)$.

The requirement that the inner product is invariant results in the following form:

$$\langle a, b\rangle_{\mathfrak{u}(\mathbb{W}_w)} = -2r_2 g'^2 \operatorname{Trace}(ab) + r_2(g'^2 - g^2) \operatorname{Trace} a \operatorname{Trace} b, \quad (68)$$

where $a, b \in \mathfrak{u}(\mathbb{W}_w)$, g, g' are constants – the coupling constants of the electroweak model, and $r_2 > 0$ is a constant.

The Weinberg angle θ_W is given by

$$\sin^2 \theta_W = \frac{g'^2}{g^2 + g'^2}, \quad (69)$$

The Weinberg Angle iv

The electric charge e is

$$e = g \sin \theta_W = g' \cos \theta_W = \frac{1}{2}\sqrt{g^2 + g'^2} \sin 2\theta_W. \tag{70}$$

The standard electroweak model does not provide a preference for this angle, which is determined indirectly from experiments.

The grand unified theories, and the present proposal, predict definite values for the Weinberg angle.

The Higgs field is a scalar with respect to spacetime symmetries, but internally it is a vector $\phi \in \mathbb{W}_W$. The direction of the vector ϕ in $\mathbb{W}_W$ is the element $\omega^{\dagger}_{\mathrm{u}} = \frac{\phi}{\sqrt{h_{\mathrm{w}}(\phi,\phi)}}$.

The Weinberg Angle V

The Higgs field has two main roles: on the one hand is responsible for the symmetry breaking, by selecting a particular direction in the space $\mathbb{W}_W$.

On the other hand, it is responsible for the masses of at least some of the elementary particles.

The Higgs field is a section of the electroweak bundle, which splits the electroweak bundle for a pair of weakly interacting leptons into two one-dimensional complex bundles – the bundle spanned by the Higgs field, and the bundle orthogonal to that.

But in the proposed approach, this split is ensured by the operator $-i\mathfrak{e}\tilde{\mathfrak{e}}$.

Recall that the representation of the Dirac algebra on one of the minimal left ideals of $\mathcal{A}_{\mathsf{SM}}$ is reducible, being eight-dimensional.

The operator $-i\mathfrak{e}\tilde{\mathfrak{e}}$ splits each ideal into two four-dimensional space by determining two projectors, $\frac{1}{2}(1 \mp i\mathfrak{e}\tilde{\mathfrak{e}})$.

Therefore, it also determines the particular direction $\omega^{\dagger}_{\mathsf{u}}$, and by this, the Higgs field ϕ up to a constant factor.

Hence, in the Standard Model Algebra, the symmetry breaking does not require the Higgs field, although it is still needed to generate the masses of the particles.

Let us now calculate the prediction of the Weinberg angle θ_W, first in general, considering an extension of $\mathfrak{u}(2)$ to $\mathfrak{su}(N)$, $2 < N \in \mathbb{N}$. I will follow a simple generalization of the usual geometric proof, used for example in (Derdzinski 1992, Ch. 7) for the SU(5) GUT.

Because $\mathfrak{su}(N)$ is simple, there is a unique Ad-invariant inner product, up to a constant r,

$$\langle A, B\rangle_{\mathrm{SU}(N)} = -Nr_N \operatorname{Trace}(AB), \tag{71}$$

where $A, B \in \mathfrak{su}(N)$, $r_N > 0$. The embedding of $\mathfrak{u}(2)$ in $\mathfrak{su}(N)$ should be traceless, because $\operatorname{Trace}(A) = 0$ for any $A \in \mathfrak{su}(N)$.

It follows that the embedding is given, in a basis extending the basis of $\mathbb{W}_w$ to $\mathbb{C}^N$. by

$$a \mapsto a \oplus \left(-\frac{1}{N-2} \operatorname{Trace} a I_{\mathbb{W}_w^\perp}\right) \tag{72}$$

for any $a \in \mathfrak{u}(2)$.

Then,

$$\begin{aligned}
&\langle a, b\rangle_{\mathfrak{u}(2)} = \\
&= \langle a \oplus \left(-\tfrac{1}{N-2} \operatorname{Trace} a I_{\mathbb{W}_w^\perp}\right), b \oplus \left(-\tfrac{1}{N-2} \operatorname{Trace} b I_{\mathbb{W}_w^\perp}\right)\rangle_{\mathrm{SU}(N)} \\
&= -N r_N \operatorname{Trace}(ab) - N r_N \left(-\tfrac{1}{N-2}\right)^2 \operatorname{Trace} a \operatorname{Trace} b \operatorname{Trace} I_{\mathbb{W}_w^\perp} \\
&= -N r_N \operatorname{Trace}(ab) - N r_N \tfrac{1}{N-2} \operatorname{Trace} a \operatorname{Trace} b.
\end{aligned} \tag{73}$$

By comparing with (68) it follows that $2 r_2 g'^2 = r_N N$ and $r_2(g'^2 - g^2) = -N r_N \frac{1}{N-2}$.

This solves to $g'^2 = \frac{N}{2}\frac{r_N}{r_2}$ and $g^2 = g'^2 + \frac{N}{N-2}\frac{r_N}{r_2}$.

Then, the Weinberg angle predicted by a GUT based on the extension of $\mathfrak{u}(2)$ to $\mathfrak{su}(N)$ is

$$\sin^2\theta_{W,N} = \frac{\frac{N}{2}}{N + \frac{N}{N-2}} = \frac{\frac{N}{2}}{\frac{N(N-1)}{N-2}} = \frac{1}{2}\frac{N-2}{N-1}. \tag{74}$$

Then, for the SU(5) GUT model one gets $\sin^2\theta_{W,5} = \frac{3}{8} = 0.375$.

For the Standard Model Algebra, recall that $\mathfrak{u}(\mathbb{W}_w)$ is embedded in $\mathfrak{su}(3)$, which is the symmetry group of $(\mathscr{N}^\dagger, h_{\mathscr{N}})$.

Then,

$$\sin^2\theta_{W,\mathcal{A}_{\mathsf{SM}}} = \sin^2\theta_{W,3} = \frac{1}{4} = 0.25, \tag{75}$$

corresponding to $\theta_{W,\mathcal{A}_{\mathsf{SM}}} = \frac{\pi}{6}$.

The prediction of $\mathcal{A}_{\mathsf{SM}}$, $\sin^2\theta_W = 0.25$, seems more encouraging that that of 0.375 predicted by the SU(5), $\mathsf{Spin}(10)$, and other GUTs.

But its derivation from the embedding of $\mathsf{U}(2)_{\mathsf{ew}}$ into an SU(3) symmetry acting on the left of the algebra $\mathcal{A}_{\mathsf{SM}}$ seems to imply an unexpected connection between the electroweak symmetry and spacetime, which requires further investigations.

Moreover, it is still not within the range estimated experimentally.

Depending on the utilized scheme, the experimental values for $\sin^2\theta_W$, range between $\sim$ 0.223 and $\sim$ 0.24 (Erler and Freitas 2015).

In particular, CODATA gives a value of 0.23129(5) (Mohr and Newe 2016). As in the case of the SU(5) prediction of $\sin^2\theta_{W,5} = 0.375$, a correct comparison would require taking into account the running of the coupling constants due to higher order perturbative corrections.

Thank you!