Highly entangled quantum spin chains

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Mainly based on

Bravyi et al, Phys. Rev. Lett. **118** (2012) 207202, arXiv: 1203.5801
R. Movassagh and P. Shor, Proc. Natl. Acad. Sci. **113** (2016) 13278, arXiv: 1408.1657
F.S. and V. Korepin, Int. J. Mod. Phys. B **32** (2018) no.28, 1850306, arXiv:1806.04049

Outline

Introduction 1

Quantum entanglement

- Most surprising feature of quantum mechanics,
  No analog in classical mechanics

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Quantum entanglement

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  No analog in classical mechanics
- From pure state of the full system S: $\rho = |\psi\rangle\langle\psi|$, reduced density matrix of a subsystem A: $\rho_A = \mathrm{Tr}_{S-A}\,\rho$ can become mixed states, and has nonzero entanglement entropy

$$S_A = -\mathrm{Tr}_A\left[\rho_A \ln \rho_A\right].$$

This is purely a quantum property.

Introduction 2

Area law of entanglement entropy

- Ground states of quantum many-body systems with local interactions typically exhibit the area law behavior of the entanglement entropy: $S_A \propto (\text{area of } A)$
- Gapped systems in 1D are proven to obey the area law.
  [Hastings 2007]

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- Belief for gapless case in D-dim. (over two decades) :
  $S_A = O(L^{D-1} \ln L)$ (L: length scale of A)
- Recently, 1D solvable spin chain models which exhibit significant area-law violation have been discovered.
  - Beyond logarithmic violation: $S_A \propto \sqrt{(\text{volume of } A)}$
    [Movassagh, Shor 2014], [Salberger, Korepin 2016]
    Counterexamples of the belief!

Motzkin spin model 1

[Bravyi et al 2012]

- 1D spin chain at sites $i \in \{1, 2, \cdots, 2n\}$
- Spin-1 state at each site can be regarded as up, down and flat steps;

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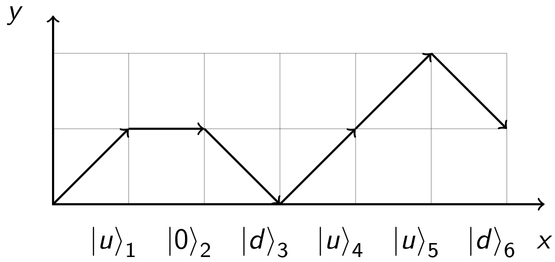
- Each spin configuration $\Leftrightarrow$ length-$2n$ walk in (x, y) plane
  Example)



Motzkin spin model 2

[Bravyi et al 2012]

Hamiltonian: $H_{Motzkin} = H_{bulk} + H_{bdy}, \quad H_{bdy} = |d\rangle_1\langle d| + |u\rangle_{2n}\langle u|$

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$$\Pi_{j,j+1} = |D\rangle_{j,j+1}\langle D| + |U\rangle_{j,j+1}\langle U| + |F\rangle_{j,j+1}\langle F|$$

(local interactions) with

$$|D\rangle \equiv \frac{1}{\sqrt{2}}\left(|0,\, d\rangle - |d,\, 0\rangle\right),$$

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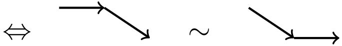

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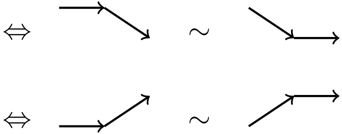

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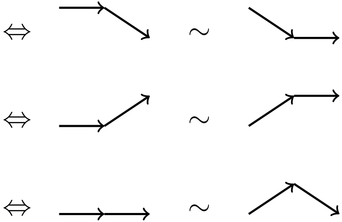

"gauge equivalence".

Motzkin spin model 3

[Bravyi et al 2012]

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$$\Downarrow$$

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- $H_{Motzkin}$ is the sum of projection operators.

  $\Rightarrow$ Positive semi-definite spectrum
- We find the unique zero-energy ground state.

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- $H_{Motzkin}$ is the sum of projection operators.

  $\Rightarrow$ Positive semi-definite spectrum
- We find the unique zero-energy ground state.
  - Each projector in $H_{Motzkin}$ annihilates the zero-energy state.

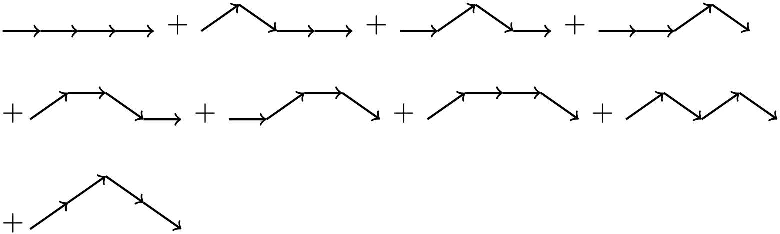
    $\Rightarrow$ Frustration free
- The ground state corresponds to randoms walks starting at $(0, 0)$ and ending at $(2n, 0)$ restricted to the region $y \geq 0$ (Motzkin Walks (MWs)).

Motzkin spin model 4

[Bravyi et al 2012]

Example) $2n = 4$ case,

MWs:



$$\Updownarrow$$

Ground state:

$$
\begin{aligned}
|P_4\rangle \quad = \quad & \frac{1}{\sqrt{9}}\left[|0000\rangle + |ud00\rangle + |0ud0\rangle + |00ud\rangle\right. \\
& +|u0d0\rangle + |0u0d\rangle + |u00d\rangle + |udud\rangle \\
& \left.+|uudd\rangle\right].
\end{aligned}
$$

Motzkin spin model 5

[Bravyi et al 2012]

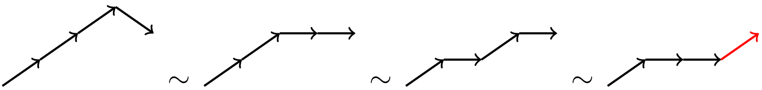
Note

Forbidden paths for the ground state

1. Path entering $y < 0$ region

Forbidden by H_{bdy}

2. Path ending at nonzero height



Forbidden by H_{bdy}

Motzkin spin model 6

[Bravyi et al 2012]

In terms of $S=1$ spin matrices

$$S_z = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}, \quad S_\pm \equiv \frac{1}{\sqrt{2}}(S_x \pm i S_y) = \begin{pmatrix} & 1 & \\ & & 1 \\ & & \end{pmatrix}, \begin{pmatrix} & & \\ 1 & & \\ & 1 & \end{pmatrix},$$

$$H_{bulk} = \frac{1}{2}\sum_{j=1}^{2n-1}\Big[1_j 1_{j+1} - \frac{1}{4}S_{z\,j}S_{z\,j+1} - \frac{1}{4}S^2_{z\,j}S_{z\,j+1} + \frac{1}{4}S_{z\,j}S^2_{z\,j+1}$$

$$-\frac{3}{4}S^2_{z\,j}S^2_{z\,j+1} + S_{+\,j}\left(S_z S_-\right)_{j+1} + S_{-\,j}\left(S_+ S_z\right)_{j+1} - \left(S_- S_z\right)_j S_{+\,j+1}$$

$$-\left(S_z S_+\right)_j S_{-\,j+1} - \left(S_- S_z\right)_j \left(S_+ S_z\right)_{j+1} - \left(S_z S_+\right)_j \left(S_z S_-\right)_{j+1}\Big],$$

$$H_{bdy} = \frac{1}{2}\left(S_z^2 - S_z\right)_1 + \frac{1}{2}\left(S_z^2 + S_z\right)_{2n}$$

Quartic spin interactions

Motzkin spin model 7

[Bravyi et al 2012]

Entanglement entropy of the subsystem $A = \{1, 2, \cdots, n\}$:

- Normalization factor of the ground state $|P_{2n}\rangle$ is given by the number of MWs of length $2n$: $M_{2n} = \sum_{k=0}^{n} C_k \binom{2n}{2k}$.

  $C_k = \frac{1}{k+1}\binom{2k}{k}$: Catalan number

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- Consider to trace out the density matrix $\rho = |P_{2n}\rangle\langle P_{2n}|$ w.r.t. the subsystem $B = \{n+1, \cdots, 2n\}$.
  Schmidt decomposition:

$$|P_{2n}\rangle = \sum_{h\geq 0} \sqrt{p_{n,n}^{(h)}} \left| P_n^{(0\to h)} \right\rangle \otimes \left| P_n^{(h\to 0)} \right\rangle$$

with $p_{n,n}^{(h)} \equiv \frac{\left(M_n^{(h)}\right)^2}{M_{2n}}$.

$\uparrow$
Paths from $(0,0)$ to (n,h)

Motzkin spin model 8

[Bravyi et al 2012]

- $M_n^{(h)}$ is the number of paths in $P_n^{(0\to h)}$.
  For $n \to \infty$, Gaussian distribution

$$p_{n,n}^{(h)} \sim \frac{3\sqrt{6}}{\sqrt{\pi}} \frac{(h+1)^2}{n^{3/2}} e^{-\frac{3}{2}\frac{(h+1)^2}{n}} \times [1 + O(1/n)] \,.$$

- Reduced density matrix

$$\rho_A = \mathrm{Tr}_B \rho = \sum_{h\geq 0} p_{n,n}^{(h)} \left| P_n^{(0\to h)} \right\rangle \left\langle P_n^{(0\to h)} \right|$$

- Entanglement entropy

$$\begin{aligned} S_A &= -\sum_{h\geq 0} p_{n,n}^{(h)} \ln p_{n,n}^{(h)} \\ &= \frac{1}{2}\ln n + \frac{1}{2}\ln\frac{2\pi}{3} + \gamma - \frac{1}{2} \qquad (\gamma\text{: Euler constant}) \end{aligned}$$

up to terms vanishing as $n \to \infty$.

Motzkin spin model 9

[Bravyi et al 2012]

Notes

- The system is critical (gapless).
  S_A is similar to the $(1+1)$-dimensional CFT.

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  The system cannot be described by relativistic CFT.
- As we will see later, the Rényi entropy exhibits different behavior from the CFT case.
- Excitations have not been much investigated.

Colored Motzkin spin model 1

[Movassagh, Shor 2014]

- Introducing color d.o.f. $k = 1, 2, \cdots, s$ to up and down spins as

$$\left|u^k\right\rangle \Leftrightarrow \overset{k}{\nearrow}, \qquad \left|d^k\right\rangle \Leftrightarrow \overset{k}{\searrow}, \qquad |0\rangle \Leftrightarrow \longrightarrow$$

Color d.o.f. decorated to Motzkin Walks

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Color d.o.f. decorated to Motzkin Walks

- Hamiltonian $H_{cMotzkin} = H_{bulk} + H_{bdy}$
  - Bulk part consisting of local interactions:

$$H_{bulk} = \sum_{j=1}^{2n-1} \left(\Pi_{j,j+1} + \Pi_{j,j+1}^{cross} \right),$$

$$\Pi_{j,j+1} = \sum_{k=1}^{s} \left[\left|D^k\right\rangle_{j,j+1} \left\langle D^k\right| + \left|U^k\right\rangle_{j,j+1} \left\langle U^k\right| + \left|F^k\right\rangle_{j,j+1} \left\langle F^k\right| \right]$$

with

Colored Motzkin spin model 2

[Movassagh, Shor 2014]

$$\left|D^k\right\rangle \equiv \frac{1}{\sqrt{2}}\left(\left|0,\, d^k\right\rangle - \left|d^k,\, 0\right\rangle\right),$$
$$\left|U^k\right\rangle \equiv \frac{1}{\sqrt{2}}\left(\left|0,\, u^k\right\rangle - \left|u^k,\, 0\right\rangle\right),$$
$$\left|F^k\right\rangle \equiv \frac{1}{\sqrt{2}}\left(\left|0,\, 0\right\rangle - \left|u^k,\, d^k\right\rangle\right),$$

and

$$\Pi^{cross}_{j,j+1} = \sum_{k\neq k'} \left|u^k,\, d^{k'}\right\rangle_{j,j+1}\left\langle u^k,\, d^{k'}\right|.$$

$\Rightarrow$ Colors should be matched in up and down pairs.

- Boundary part

$$H_{bdy} = \sum_{k=1}^{s}\left(\left|d^k\right\rangle_1\left\langle d^k\right| + \left|u^k\right\rangle_{2n}\left\langle u^k\right|\right).$$

Colored Motzkin spin model 3

[Movassagh, Shor 2014]

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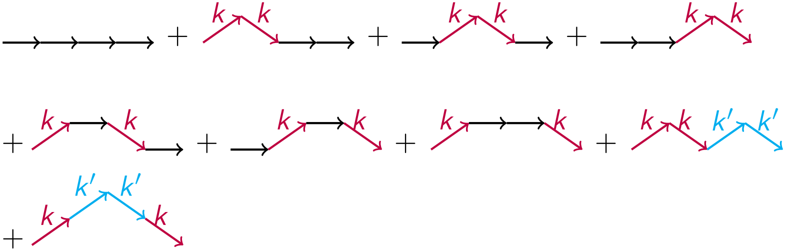


$$|P_4\rangle = \frac{1}{\sqrt{1+6s+2s^2}}\left[|0000\rangle + \sum_{k=1}^{s}\left\{\left|u^k d^k 00\right\rangle + \cdots + \left|u^k 00 d^k\right\rangle\right\}\right.$$

$$\left. + \sum_{k,k'=1}^{s}\left\{\left|u^k d^k u^{k'} d^{k'}\right\rangle + \left|u^k u^{k'} d^{k'} d^k\right\rangle\right\}\right].$$

Colored Motzkin spin model 4

[Movassagh, Shor 2014]

Entanglement entropy

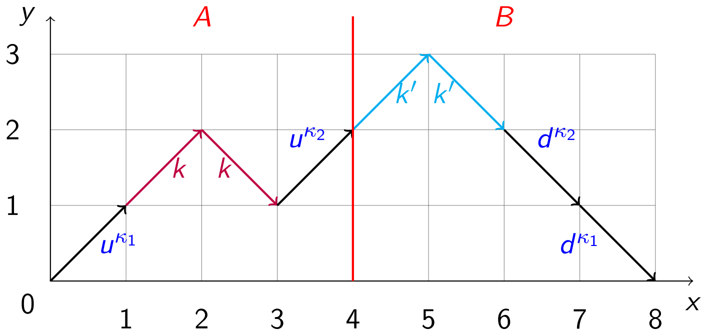
- Paths from $(0,0)$ to (n,h), $P_n^{(0\to h)}$, have h unmatched up steps.
  Let $\tilde{P}_n^{(0\to h)}(\{\kappa_m\})$ be paths with the colors of unmatched up steps frozen.
  (unmatched up from height $(m-1)$ to m) $\to u^{\kappa_m}$
- Similarly,
  $$P_n^{(h\to 0)} \to \tilde{P}_n^{(h\to 0)}(\{\kappa_m\}),$$
  (unmatched down from height m to $(m-1)$) $\to d^{\kappa_m}$.
- The numbers satisfy $M_n^{(h)} = s^h\, \tilde{M}_n^{(h)}$.

Colored Motzkin spin model 5

[Movassagh, Shor 2014]

Example

$2n = 8$ case, $h = 2$



Colored Motzkin spin model 6

[Movassagh, Shor 2014]

- Schmidt decomposition

$$
\begin{aligned}
|P_{2n}\rangle &= \sum_{h\geq 0}\sum_{\kappa_1=1}^{s}\cdots\sum_{\kappa_h=1}^{s}\sqrt{p_{n,n}^{(h)}} \\
&\times \left|\tilde{P}_n^{(0\to h)}(\{\kappa_m\})\right\rangle \otimes \left|\tilde{P}_n^{(h\to 0)}(\{\kappa_m\})\right\rangle
\end{aligned}
$$

with

$$
p_{n,n}^{(h)} = \frac{\left(\tilde{M}_n^{(h)}\right)^2}{M_{2n}}.
$$

- Reduced density matrix

$$
\begin{aligned}
\rho_A &= \sum_{h\geq 0}\sum_{\kappa_1=1}^{s}\cdots\sum_{\kappa_h=1}^{s} p_{n,n}^{(h)} \\
&\times \left|\tilde{P}_n^{(0\to h)}(\{\kappa_m\})\right\rangle\left\langle\tilde{P}_n^{(0\to h)}(\{\kappa_m\})\right|.
\end{aligned}
$$

Colored Motzkin spin model 7

[Movassagh, Shor 2014]

- For $n \to \infty$,

$$p_{n,n}^{(h)} \sim \frac{\sqrt{2}\, s^{-h}}{\sqrt{\pi}\,(\sigma n)^{3/2}}\,(h+1)^2\, e^{-\frac{(h+1)^2}{2\sigma n}} \times [1 + O(1/n)]$$

with $\sigma \equiv \frac{\sqrt{s}}{2\sqrt{s}+1}$. Note: Effectively $h \lesssim O(\sqrt{n})$.

- Entanglement entropy

$$S_A \;=\; -\sum_{h\geq 0} s^h\, p_{n,n}^{(h)} \ln p_{n,n}^{(h)}$$

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- Entanglement entropy

$$\begin{aligned} S_A &= -\sum_{h\geq 0} s^h\, p_{n,n}^{(h)} \ln p_{n,n}^{(h)} \\ &= (2\ln s)\sqrt{\frac{2\sigma\, n}{\pi}} + \frac{1}{2}\ln n + \frac{1}{2}\ln(2\pi\sigma) + \gamma - \frac{1}{2} - \ln s \end{aligned}$$

up to terms vanishing as $n \to \infty$.

Grows as $\sqrt{n}$.

Colored Motzkin spin model 8

[Movassagh, Shor 2014]

Comments

- $$\begin{aligned} \text{Matching color} \quad &\Rightarrow \quad {\color{red} s^{-h}} \text{ factor in } p_{n,n}^{(h)} \\ &\Rightarrow \quad \text{crucial to } O(\sqrt{n}) \text{ behavior in } S_A \end{aligned}$$

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Comments

- Matching color $\Rightarrow$ s^{-h} factor in $p_{n,n}^{(h)}$
  $\Rightarrow$ crucial to $O(\sqrt{n})$ behavior in S_A
- For spin $1/2$ chain (only up and down), the model in which similar behavior exhibits in colored as well as uncolored cases has been constructed. (Fredkin model) [Salberger, Korepin 2016]

Colored Motzkin spin model 8 [Movassagh, Shor 2014]

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- For spin 1/2 chain (only up and down), the model in which similar behavior exhibits in colored as well as uncolored cases has been constructed. (Fredkin model) [Salberger, Korepin 2016]
- Deformation of models to achieve the volume law behavior ($S_A \propto n$)
  Weighted Motzkin/Dyck walks [Zhang et al, Salberger et al 2016]

Rényi entropy

[Rényi, 1970]

- Rényi entropy has further importance than the von Neumann entanglement entropy:

$$S_{A,\alpha} = \frac{1}{1-\alpha} \ln \mathrm{Tr}_A \, \rho_A^{\alpha} \qquad \text{with } \alpha > 0 \text{ and } \alpha \neq 1.$$

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- Generalization of the von Neumann entanglement entropy: $\lim_{\alpha\to 1} S_{A,\alpha} = S_A$

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- Generalization of the von Neumann entanglement entropy: $\lim_{\alpha\to 1} S_{A,\alpha} = S_A$
- Reconstructs the whole spectrum of the entanglement Hamiltonian $H_{\mathrm{ent},A} \equiv -\ln \rho_A$.

Rényi entropy

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- Rényi entropy has further importance than the von Neumann entanglement entropy:

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Here, we analytically compute the Rényi entropy of half-chain in the Motzkin model.

New phase transition found at $\alpha = 1$!

Réyni entropy of Motzkin model 1

[F.S., Korepin, 2018]

- What we compute is the asymptotic behavior of

$$S_{A,\alpha} = \frac{1}{1-\alpha} \ln \sum_{h=0}^{n} s^h \left(p_{n,n}^{(h)} \right)^{\alpha} .$$

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$$\begin{aligned} S_{A,\alpha} &= \frac{1}{2}\ln n + \frac{1}{1-\alpha}\ln\Gamma\left(\alpha+\frac{1}{2}\right) \\ &\quad -\frac{1}{2(1-\alpha)}\left\{(1+2\alpha)\ln\alpha + \alpha\ln\frac{\pi}{24} + \ln 6\right\} \end{aligned}$$

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up to terms vanishing as $n \to \infty$.

  - Logarithmic growth, but different from the CFT case
  - Reduces to S_A in the $\alpha \to 1$ limit.
  - Consistent with half-chain case in the result in [Movassagh, 2017]

Réyni entropy of Motzkin model 2 [F.S., Korepin, 2018]

Colored case ($s > 1$)

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$$S_{A,\alpha} = n\frac{2\alpha}{1-\alpha} \ln\left[\sigma\left(s^{\frac{1-\alpha}{2\alpha}} + s^{-\frac{1-\alpha}{2\alpha}} + s^{-1/2}\right)\right] + \frac{1+\alpha}{2(1-\alpha)} \ln n + C(s,\alpha)$$

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  - Note: $\alpha \to 1$ or $s \to 1$ limit does not commute with the $n \to \infty$ limit.

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[F.S., Korepin, 2018]

Rényi entropy for $\alpha > 1$

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Phase transition

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- The transition point $\alpha = 1$ itself forms the third phase.

$S_{A,\alpha}$: $O(\ln n)$ $O(\sqrt{n})$ $O(n)$

0 1 $1/\alpha$

h: $O(n^0)$ $O(\sqrt{n})$ $O(n)$

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[F.S., Padmanabhan, 2018; Padmanabhan, F.S., Korepin, 2018]

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  [F.S., Padmanabhan, 2018; Padmanabhan, F.S., Korepin, 2018]
- As a feature of the extended models,
  Anderson-like localization occurs in excited states corresponding to disconnected paths.
  - "2nd quantized paths".

Summary and discussion 2

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  - The same phase transition occurs for chain of general length
  - In the deformed Fredkin model with $s > 1$ and $t > 1$, such phase transition does not happen. [Udagawa, Katsura 2017]

Summary and discussion 3

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  [Personal speculation]

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Thank you very much for your attention!