Conformal theory of MacDowell-Mansouri type

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MacDowell–Mansouri action

$$S_{MM} = \frac{3}{4G\Lambda}\int_M F^{IJ}\wedge F^{KL}\epsilon_{IJKL} \tag{1}$$

It is possible to show that S_{MM} is equivalent to Palatini action.

$$S_{Pal} = \frac{1}{2G}\int_M (e^i\wedge e^j\wedge R^{kl} - \frac{\Lambda}{6}e^i\wedge e^j\wedge e^k\wedge e^l)\epsilon_{ijkl} \tag{2}$$

BF action

$$S = \int_{M^4} \boxed{B^{IJ} \wedge F_{IJ}} - \boxed{\frac{\beta}{2} B^{IJ} \wedge B_{IJ}} - \boxed{\frac{1}{2}\epsilon_{IJKLM} v^M \wedge B^{IJ} \wedge B^{KL}}. \tag{3}$$

where $I, J = 0 \dots 4$.

$$v^M = \left(0 \cdots \frac{\alpha}{2}\right)^T$$

where $\alpha \approx 10^{-120}$

BF action. Motivation

The construction base on two term. One is a topological term which generates the topological vacuum and second term are breaking symmetry down to Lorentz symmetry.

This kind of construction has 3 adventiges:

1. The Lagrangian is quadratic in fields, where the Palatini formalism is trilinear.
2. The presented modification will introduce Immirzi parameter.
3. It allows for introducing dynamical degrees of freedom as a perturbation around topological vacuum.

BF action conti.

$$S = \int_{M^4} B^{ij} \wedge F_{ij} - \frac{\beta}{2} B^{ij} \wedge B_{ij} - \frac{\alpha}{2} \epsilon_{ijkl} \wedge B^{ij} \wedge B^{kl}. \tag{4}$$

where $i, j, \dots = 0 \dots 3$.
Note that $\alpha \neq \beta$. If $\alpha = \beta$ then it leads to self–dual gravity. The physical meaning of constatnts is:

$$\boxed{\frac{1}{\ell^2} = \frac{\Lambda}{3}} \qquad \boxed{\alpha = \frac{\boxed{G\Lambda}}{3(1-\gamma^2)}} \qquad \boxed{\beta = \frac{\boxed{G\Lambda}\,\gamma}{3(1-\gamma^2)}} \tag{5}$$

BF action conti.

$$S = S_{H+\Lambda} + \int_{M^4} \Big(\frac{\alpha}{4(\alpha^2 - \beta^2)} \boxed{E(\omega)} - \frac{\beta}{2(\alpha^2 - \beta^2)} \boxed{P(\omega)} + \frac{1}{\beta} \boxed{NY(\omega, e)} \Big) \tag{6}$$

where

$$S_{H+\Lambda} = -\frac{1}{G}\epsilon_{ijkl}(R^{ij} \wedge e^k \wedge e^l - \frac{\Lambda}{3} e^i \wedge e^j \wedge e^k \wedge e^l) - \frac{2}{G\gamma} R^{ij} \wedge e_i \wedge e_j \tag{7}$$

and $E(\omega)$ is Euler $P(\omega)$ is Pontryagin $NY(\omega, e)$ in Nieh-Yan invariants.

Conformal algebra

$$[\mathcal{D}, \mathcal{K}_i] = i\mathcal{K}_i \,, \tag{8}$$

$$[\mathcal{D}, \mathcal{P}_i] = -i\mathcal{P}_i \,, \tag{9}$$

$$[\mathcal{K}_i, \mathcal{P}_j] = -2i(\eta_{ij}\mathcal{D} - \mathcal{M}_{ij}) \,, \tag{10}$$

$$\boxed{[\mathcal{M}_{ij}, \mathcal{K}_k] = -i(\eta_{ki}\mathcal{K}_j - \eta_{kj}\mathcal{K}_i)} \,, \tag{11}$$

$$\boxed{[\mathcal{M}_{ij}, \mathcal{P}_k] = -i(\eta_{ki}\mathcal{P}_j - \eta_{kj}\mathcal{P}_i)} \,, \tag{12}$$

$$\boxed{[\mathcal{M}_{ij}, \mathcal{M}_{kl}] = -i(\eta_{ik}\mathcal{M}_{jl} + \eta_{jl}\mathcal{M}_{ik} - \eta_{jk}\mathcal{M}_{il} - \eta_{il}\mathcal{M}_{jk})} \,, \tag{13}$$

Initial action

$$S = \int_{M^4} \boxed{B^{IJ} \wedge F_{IJ}} - \boxed{\frac{\beta}{2} B^{IJ} \wedge B_{IJ}} - \boxed{\frac{1}{2} \epsilon_{IJKLMN} v^{MN} \wedge B^{IJ} \wedge B^{KL}} . \tag{14}$$

where $I, J = 0 \dots 5$.

$$v^{MN} = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \frac{\alpha}{2} \\ 0 & \cdots & -\frac{\alpha}{2} & 0 \end{pmatrix}$$

where $\alpha \approx 10^{-120}$

Initial action conti.

$$S = \int_{M^4} \frac{1}{2}(M_{ijkl})^{-1} F^{ij} \wedge F^{kl} + \frac{1}{\beta}(F^{i4} \wedge F_{i4} - F^{i5} \wedge F_{i5} - F^{45} \wedge F_{45}) \tag{15}$$

where $M_{ij}{}^{kl} = (\beta \delta_{ij}{}^{kl} + \frac{\alpha}{2} \epsilon_{ij}{}^{kl})$

$$F^{ij} M_{ij} = (R^{ij} - \frac{1}{\ell^2} f_1^i \wedge f_1^j + \frac{1}{\ell^2} f_2^i \wedge f_2^j) M_{ij} \tag{16}$$

$$F^{45} D = (\frac{1}{\ell} d\phi - \frac{1}{\ell^2} f_1^j \wedge f_{2j}) D \tag{17}$$

$$F^{i4} R_{1i} = (\frac{1}{\ell} D^\omega f_1^i - \frac{1}{\ell^2} \phi \wedge f_2^i) R_{1i} \tag{18}$$

$$F^{i4} R_{2i} = (\frac{1}{\ell} D^\omega f_2^i - \frac{1}{\ell^2} \phi \wedge f_1^i) R_{2i} \,. \tag{19}$$

substitution...

$$
\begin{aligned}
S = \int_{\mathcal{M}^4} \frac{1}{4}\Big(\frac{\alpha}{\alpha^2+\beta^2}\big(\frac{\beta}{\alpha}\delta_{ijkl} - \epsilon_{ijkl}\big)\Big)\Big(R^{ij}\wedge R^{kl} - \frac{2}{\ell^2}R^{ij}\wedge f_1^k \wedge f_1^l \\
+ \frac{2}{\ell^2}R^{ij}\wedge f_2^k \wedge f_2^l + \frac{1}{\ell^4} f_1^i \wedge f_1^j \wedge f_1^k \wedge f_1^l \\
+ \frac{1}{\ell^4} f_2^i \wedge f_2^j \wedge f_2^k \wedge f_2^l - \frac{2}{\ell^4} f_1^i \wedge f_1^j \wedge f_2^k \wedge f_2^l\Big) \qquad (20)
\end{aligned}
$$

$$
\begin{aligned}
- \frac{1}{\beta\ell^2}\Big(d\phi \wedge d\phi - \frac{2}{\ell} d\phi \wedge f_1^i \wedge f_{2i} - \frac{1}{\ell^2} f_1^i \wedge f_{2i} \wedge f_1^j \wedge f_{2j}\Big) \\
+ \frac{1}{\beta\ell^2}\Big(D^\omega f_1^i \wedge D^\omega f_{1i} - \frac{2}{l} D^\omega f_1^i \wedge \phi \wedge f_{2i}\Big) \\
- \frac{1}{\beta\ell^2}\Big(D^\omega f_2^i \wedge D^\omega f_{2i} - \frac{2}{l} D^\omega f_2^i \wedge \phi \wedge f_{1i}\Big)\,. \qquad (21)
\end{aligned}
$$

The field equations are

$$\delta\phi \quad : \frac{2}{\beta\ell^3}(D^\omega f_1^a \wedge f_{2a} - D^\omega f_2^a \wedge f_{1a} - d(f_1^a \wedge f_{2a})) = 0 \tag{22}$$

$$\delta f_1^a + \delta f_2^a \; : D^\omega(\frac{1}{2}\epsilon_{abcd} e^c \wedge e^d + \frac{1}{\gamma} e_a \wedge e_b) = D^\omega(\frac{1}{2}\epsilon_{abcd} f^c \wedge f^d + \frac{1}{\gamma} f_a \wedge f_b) \tag{23}$$

$$\delta\omega^{ab} : \; - \frac{1}{2G}\epsilon_{abcd} D^\omega(f_2^c \wedge f_2^d) + \frac{1}{2G}\epsilon_{abcd} D^\omega(f_1^c \wedge f_1^d) \tag{24}$$

$$- \frac{1}{\gamma G} D^\omega(f_{2a} \wedge f_{2b}) + \frac{1}{\gamma G} D^\omega(f_{1a} \wedge f_{1b}) \tag{25}$$

$$+ \frac{2}{\beta\ell^2} f_{1[b} \wedge f_{2a]} \wedge \phi - \frac{2}{\beta\ell^2} f_{2[b} \wedge f_{1a]} \wedge \phi = 0\,. \tag{26}$$

Changing algebra basie

At that point it is more convenient to use a conformal groups isomorphism and to change the algebra base. Notice that by such treatment the vector field transform as follows

$$f_1^i = \frac{1}{2}(e^i - f^i) \tag{27}$$

$$f_2^i = \frac{1}{2}(e^i + f^i)\,. \tag{28}$$

$$\frac{1}{G}\epsilon_{ijkl}R^{ij} \wedge e^k \wedge f^l + \frac{1}{2\ell^2 G}\epsilon_{ijkl}e^i \wedge f^j \wedge e^k \wedge f^l = 0\,. \tag{29}$$

$$D^\omega(\frac{1}{2}\epsilon_{abcd}e^c \wedge e^d + \frac{1}{\gamma}e_a \wedge e_b) = D^\omega(\frac{1}{2}\epsilon_{abcd}f^c \wedge f^d + \frac{1}{\gamma}f_a \wedge f_b)\,. \tag{30}$$

Action

$$
\begin{aligned}
L = & -\frac{1}{2G}\epsilon_{ijkl}R^{ij}\wedge e^k \wedge f^l - \frac{1}{G\gamma}R^{ij}\wedge e_i \wedge f_j \\
& -\frac{1}{4\ell^2 G}\epsilon_{ijkl}e^i \wedge e^j \wedge f^k \wedge f^l \\
& -\frac{1}{\beta\ell^2}S_4(\phi) + \frac{1}{\beta\ell^3}C_4(e,f,\phi) + \frac{\ell^2}{4G}E_4(\omega) \\
& +\frac{\gamma\ell^2}{2G}P_4(\omega) + 2\frac{\gamma^2+1}{\gamma G}NY_4(e,f) \ ,
\end{aligned}
\tag{31}
$$

Action

$$
\begin{aligned}
L = & -\frac{1}{2G}\epsilon_{ijkl}R^{ij}\wedge e^k \wedge f^l - \frac{1}{G\gamma}R^{ij}\wedge e_i \wedge f_j \\
& -\frac{1}{4\ell^2 G}\epsilon_{ijkl}e^i \wedge e^j \wedge f^k \wedge f^l \\
& -\frac{1}{\beta\ell^2}S_4(\phi) + \frac{1}{\beta\ell^3}C_4(e,f,\phi) + \frac{\ell^2}{4G}E_4(\omega) \\
& +\frac{\gamma\ell^2}{2G}P_4(\omega) + 2\frac{\gamma^2+1}{\gamma G}NY_4(e,f) \ ,
\end{aligned}
\tag{32}
$$

Topological invariants

$$E_4(\omega) = \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu ij} \star R_{\alpha\beta}{}^{ij} \tag{33}$$

$$P_4(\omega) = \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu ij} R_{\alpha\beta}{}^{ij} \tag{34}$$

$$NY_4(e, f) = \frac{1}{2} NY_4(e - f) = \frac{1}{2} \partial_\mu [\epsilon^{\mu\nu\alpha\beta} (e - f)_\nu{}^I \cdot D_\alpha (e - f)_{\beta I}] \tag{35}$$

$$S_4(\phi) = d(\phi \wedge d\phi) \tag{36}$$

$$C_4(e, f, \phi) = d(f^a \wedge e_a \wedge \phi) \tag{37}$$

Topological invariants

In CS notation:

$$\boxed{E_4(\omega) = \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu ij} \star R_{\alpha\beta}{}^{ij}} \tag{38}$$

$$\boxed{P_4(\omega) = \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu ij} R_{\alpha\beta}{}^{ij}} \tag{39}$$

$$\boxed{NY_4(e,f) = \frac{1}{2} NY_4(e-f) = \frac{1}{2} \partial_\mu [\epsilon^{\mu\nu\alpha\beta} (e-f)_\nu{}^I \cdot D_\alpha (e-f)_{\beta I}]} \tag{40}$$

$$\boxed{S_4(\phi) = d(\phi \wedge d\phi)} \, , \tag{41}$$

$$C_4(e,f,\phi) = d(f^a \wedge e_a \wedge \phi) \tag{42}$$

Topological invariants in CS notation

$$\boxed{E_4(\omega) = 32id(C({}^{+}\omega) + C({}^{-}\omega))} \tag{43}$$

$$\boxed{P_4(\omega) = 16d(C({}^{+}\omega) + C({}^{-}\omega))} \tag{44}$$

$$\boxed{NY_4(e,f) = \frac{1}{2}dC(e-f)} \tag{45}$$

$$\boxed{S_4(\phi) = dC(\phi)} \tag{46}$$

$$C_4(e,f,\phi) = d(-6R^a \wedge e_a \wedge \phi + R \wedge \phi)\,. \tag{47}$$

Constraints and distingues frame field

If one distinguishes e^a as a field associated with translation generator, the solution of is[1]

$$f^a_\mu = -6R^a_\mu + Re^a_\mu\,. \tag{48}$$

The above assumption can be justify if the constraint[2]

$$D^\omega(e^c) = 0 \Leftrightarrow D^\omega(f^c) = 0 \tag{49}$$

has been added. Then, indeed ω becomes to be a spin connection field e.i. $\omega = \omega(e, \phi)$.

[1] M. Kaku, P. K. Townsend and P. van Nieuwenhuizen, Phys. Rev. D **17** (1978) 3179.
[2] P. K. Townsend and P. van Nieuwenhuizen, Phys. Rev. D **19** (1979) 3166.

Ation as a function of 'graviton'

$$
\begin{aligned}
L = & -\frac{1}{2G} const \cdot e \cdot (R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2) + \boxed{L^{Holst}} \\
& - \frac{1}{\beta \ell^2} S_4(\phi) + \frac{1}{\beta \ell^3} C_4(e, \phi) + \frac{\ell^2}{4G} E_4(\omega) \\
& + \frac{\gamma \ell^2}{2G} P_4(\omega) + 2 \frac{\gamma^2 + 1}{\gamma G} N Y_4(e) \, .
\end{aligned} \tag{50}
$$

where

$$
\boxed{L^{Holst}(\omega, e) = \frac{1}{G\gamma} R^{ij} \wedge e_i \wedge f_j \;\; = \frac{24 * 5}{G\gamma} (\star R) \wedge R \, .} \tag{51}
$$

(52)

Topological terms conti.

$$L = -\frac{1}{4G} const \cdot e \cdot (R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 2R^{\mu\nu} R_{\mu\nu} + \frac{1}{3} R^2) + L^{Holst}$$
$$\boxed{-\frac{1}{4G} const \cdot e \cdot GB_4(\omega, e)} - \frac{1}{\beta \ell^2} S_4(\phi) + \frac{1}{\beta \ell^3} C_4(e, \omega, \phi)$$
$$+ \frac{\ell^2}{4G} E_4(\omega) + \frac{\gamma \ell^2}{2G} P_4(\omega) + 2\frac{\gamma^2 + 1}{\gamma G} NY_4(e, \omega)\,. \quad (53)$$

The Gauss-Bonet term

$$\boxed{GB_4(\omega, e) = -(R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R^2)} \quad (54)$$

Weyl action

$$L = -\frac{1}{4G} const \cdot \boxed{L^{Weyl}} + L^{Holst} + \text{“}topological\ terms\text{”}\ . \tag{55}$$

where

$$\boxed{L^{Weyl} = C_{\mu\nu ab} C_{\rho\sigma cd} \epsilon^{\mu\nu\rho\sigma} \epsilon^{abcd} e = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 2R^{\mu\nu} R_{\mu\nu} + \frac{1}{3} R^2 \cdot e} \tag{56}$$

Thank you for your attention!

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