Starobinsky cosmological model in Palatini formalism

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10th MATHEMATICAL PHYSICS MEETING
Belgrade, 9-14 September 2019

Abstract

We consider the Starobinsky model $f(\hat{R}) = \hat{R} + \gamma\hat{R}^2$ in the Palatini formalism in both Jordan and Einstein frames. The dynamics of models is also studied using dynamical system methods. We show the evolution of the Friedmann equation can be reduced to the form of a piecewise smooth dynamical system. In result, this system is reduced to a 2D dynamical system of the Newtonian type. From the phase portraits, one can find generic evolutionary scenarios of the evolution of the Universe. At each frame the topological structures of the phase space are different. In the Jordan frame, the sewn singularity appears which represents a finite scale factor type. Such singularity appears in the Starobinsky model in the Palatini formalism when dynamics is determined by the corresponding piecewise-smooth dynamical system. After reformulation of the model in the Einstein frame, we get the FRW cosmological model with a homogeneous scalar field and the vanishing kinetic energy term.

In the Einstein frame, in the Friedmann equation, dark energy is in the form of a scalar field with a potential whose the form is determined in a covariant way by the Ricci scalar of the FRW metric. In this frame, the energy density of matter and dark energy are also parameterized through the Ricci scalar and an interaction appears between matter and dark energy because the dark energy is decaying. In this model, during the cosmic evolution, the accelerating phase for the late times and the early inflation exist. In the Einstein frame undesirable singularities disappear. We calculate the slow roll parameters and the constant roll parameter in terms of the Ricci scalar for the characterization of inflation. We have found a characteristic behavior of the time dependence of density of dark energy on the cosmic time following the logistic-like curve which interpolates two almost constant value phases. From the required numbers of e-folds N we found a limit on the model parameter. These models in both frames are also analysed by statistical methods.

Organization

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Standard Cosmological Model (ΛCDM)

The action for gravity is introduced as

$$S = S_{\mathrm{g}} + S_{\mathrm{m}} = \frac{1}{2}\int \sqrt{-g}Rd^4x + S_{\mathrm{m}}, \tag{1}$$

where $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar and $R_{\mu\nu}$ is the Ricci tensor. We assume that $8\pi G = c = 1$.

Standard Cosmological Model (ΛCDM)

In cosmology, the ΛCDM model has a status of the standard cosmological model. In this case, we assume that the metric g is FRW metric

$$ds^2 = dt^2 - a^2(t)\left[\frac{1}{1-kr^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{2}$$

where $a(t)$ is the scale factor, k is a constant of spatial curvature ($k = 0, \pm 1$), t is the cosmological time. The cosmological equations for this model are the following

$$3H^2 = (\rho + \Lambda) - \frac{k}{a^2}, \tag{3}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p) + \frac{\Lambda}{3} \tag{4}$$

and

$$\dot{\rho} = -3H(\rho + p), \tag{5}$$

where $H = \frac{\dot{a}}{a}$.

Standard Cosmological Model (ΛCDM)

As a source of gravity we assume perfect fluid with the energy-momentum tensor

$$T^{\mu}_{\nu} = \mathrm{diag}(-\rho, p, p, p), \tag{6}$$

where $p = w\rho$, $w = const$ is a form of the equation of state ($w = 0$ for dust and $w = 1/3$ for radiation). Formally, effects of the spatial curvature can be also included to the model by introducing a curvature fluid $\rho_{\mathrm{k}} = -\frac{k}{2}a^{-2}$, with the barotropic factor $w = -\frac{1}{3}$ ($p_{\mathrm{k}} = -\frac{1}{3}\rho_{\mathrm{k}}$). From the conservation condition $T^{\mu}_{\nu;\mu} = 0$ we obtain that $\rho = \rho_0 a^{-3(1+w)}$. Therefore trace T reads as

$$T = \sum_i \rho_{i,0}(3w_i - 1)a(t)^{-3(1+w_i)}. \tag{7}$$

In what follows we consider visible and dark matter ρ_{m} in the form of dust $w = 0$, dark energy ρ_{Λ} with $w = -1$ and radiation ρ_{r} with $w = 1/3$.

Palatini formalism in cosmology

In the Palatini gravity action for $f(\hat{R})$ gravity is introduced to be

$$S = S_{\mathrm{g}} + S_{\mathrm{m}} = \frac{1}{2}\int \sqrt{-g} f(\hat{R}) d^4x + S_{\mathrm{m}}, \tag{8}$$

where $\hat{R} = g^{\mu\nu}\hat{R}_{\mu\nu}(\Gamma)$ is the generalized Ricci scalar and $\hat{R}_{\mu\nu}(\Gamma)$ is the Ricci tensor of a torsionless connection Γ. We assume that $8\pi G = c = 1$. The equation of motion obtained from the first order Palatini formalism reduces to

$$f'(\hat{R})\hat{R}_{\mu\nu} - \frac{1}{2} f(\hat{R}) g_{\mu\nu} = T_{\mu\nu}, \tag{9}$$

$$\hat{\nabla}_\alpha(\sqrt{-g} f'(\hat{R}) g^{\mu\nu}) = 0, \tag{10}$$

where $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta L_{\mathrm{m}}}{\delta g_{\mu\nu}}$ is energy momentum tensor, i.e. one assumes that matter couples to the metric. In eq. (10) $\hat{\nabla}_\alpha$ means the covariant derivative calculated with respect to Γ.

Palatini formalism in cosmology

Taking the trace of (9), we obtain additional so called structural equation

$$f'(\hat{R})\hat{R} - 2f(\hat{R}) = T. \tag{11}$$

where $T = g^{\mu\nu} T_{\mu\nu}$.

Starobinsky model in Palatini formalism – Jordan frame

Because a form of the function $f(\hat{R})$ is unknown, one needs to probe it via ensuing cosmological models. Here we choose the simplest modification of the general relativity Lagrangian

$$f(\hat{R}) = \hat{R} + \gamma\hat{R}^2, \tag{12}$$

induced by first three terms in the power series decomposition of an arbitrary function $f(R)$. In fact, since the terms $\hat{R}^n$ have different physical dimensions, i.e. $[\hat{R}^n] \neq [\hat{R}^m]$ for $n \neq m$, one should take instead the function $\hat{R}_0 f(\hat{R}/\hat{R}_0)$ for constructing our Lagrangian, where $\hat{R}_0$ is a constant and $[\hat{R}_0] = [\hat{R}]$. In this case the power series expansion reads: $\hat{R}_0 f(\hat{R}/\hat{R}_0) = \hat{R}_0 \sum_{n=0} \alpha_n (\hat{R}/\hat{R}_0)^n = \sum_{n=0} \tilde{\alpha}_n \hat{R}^n$, where the coefficients α_n are dimensionless, while $[\tilde{\alpha}_n] = [\hat{R}]^{1-n}$ are dimension full.

Starobinsky model in Palatini formalism – Jordan frame

From the other hand the Lagrangian (12) can be viewed as a simplest deviation, by the quadratic Starobinsky term, from the Lagrangian $\hat{R}$ which provides the standard cosmological model a.k.a. ΛCDM model. A corresponding solution of the structural equation (11) is in the following form

$$\hat{R} = -T \equiv 4\rho_{\Lambda,0} + \rho_{\mathrm{m},0} a^{-3}. \tag{13}$$

The Friedmann equation

$$H^2 = \frac{1}{3}\left(\rho_{\mathrm{r},0} a^{-4} + \rho_{\mathrm{m},0} a^{-3} + \rho_{\Lambda,0}\right) \tag{14}$$

for the ΛCDM model in our model is modified.

Palatini formalism in cosmology

A counterpart of the above formula in our extended model can be presented as follows

$$\frac{H^2}{H_0^2} = \frac{b^2}{\left(b+\frac{d}{2}\right)^2}\Bigg(\Omega_\gamma(\Omega_{\text{m},0}a^{-3}+\Omega_{\Lambda,0})^2\frac{(K-3)(K+1)}{2b} + (\Omega_{\text{m},0}a^{-3}+\Omega_{\Lambda,0}) + \frac{\Omega_{\text{r},0}a^{-4}}{b} + \Omega_k\Bigg), \quad (15)$$

where

$$\Omega_k = -\frac{k}{H_0^2 a^2}, \tag{16}$$

$$\Omega_{\mathrm{r},0} = \frac{\rho_{\mathrm{r},0}}{3H_0^2}, \tag{17}$$

$$\Omega_{\mathrm{m},0} = \frac{\rho_{\mathrm{m},0}}{3H_0^2}, \tag{18}$$

$$\Omega_{\Lambda,0} = \frac{\rho_{\Lambda,0}}{3H_0^2}, \tag{19}$$

$$K = \frac{3\Omega_{\Lambda,0}}{(\Omega_{\mathrm{m},0}a^{-3} + \Omega_{\Lambda,0})}, \tag{20}$$

$$\Omega_\gamma = 3\gamma H_0^2, \tag{21}$$

$$b = f'(\hat{R}) = 1 + 2\Omega_\gamma(\Omega_{\mathrm{m},0}a^{-3} + 4\Omega_{\Lambda,0}), \tag{22}$$

$$d = \frac{1}{H}\frac{db}{dt} = -2\Omega_\gamma(\Omega_{\mathrm{m},0}a^{-3} + \Omega_{\Lambda,0})(3 - K). \tag{23}$$

Starobinsky model in Palatini formalism – Einstein frame

The action (8) is dynamically equivalent to the first order Palatini gravitational action, provided that $f''(\hat{R}) \neq 0$

$$S(g_{\mu\nu}, \Gamma^{\lambda}_{\rho\sigma}, \chi) = \frac{1}{2} \int \mathrm{d}^4x \sqrt{-g} \left(f'(\chi)(\hat{R} - \chi) + f(\chi) \right) + S_m(g_{\mu\nu}, \psi), \quad (24)$$

Let a scalar field $\Phi = f'(\chi)$ and $\chi = \hat{R}$, then action (24) has the following form

$$S(g_{\mu\nu}, \Gamma^{\lambda}_{\rho\sigma}, \Phi) = \frac{1}{2} \int \mathrm{d}^4x \sqrt{-g} \left(\Phi \hat{R} - U(\Phi) \right) + S_m(g_{\mu\nu}, \psi), \qquad (25)$$

where the potential $U(\Phi)$ is given by

$$U_f(\Phi) \equiv U(\Phi) = \chi(\Phi)\Phi - f(\chi(\Phi)) \qquad (26)$$

and $\Phi = \frac{df(\chi)}{d\chi}$ and $\hat{R} \equiv \chi = \frac{dU(\Phi)}{d\Phi}$.

The Palatini variation of action (25) gives us

$$\Phi\left(\hat{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\hat{R}\right) + \frac{1}{2}g_{\mu\nu}U(\Phi) - T_{\mu\nu} = 0, \tag{27a}$$

$$\hat{\nabla}_\lambda(\sqrt{-g}\Phi g^{\mu\nu}) = 0, \tag{27b}$$

$$\hat{R} - U'(\Phi) = 0. \tag{27c}$$

Equation (27b) implies that the connection $\hat{\Gamma}$ is a metric connection for the new metric $\bar{g}_{\mu\nu} = \Phi g_{\mu\nu}$. The g-trace of (27a) gives a new structural equation

$$2U(\Phi) - U'(\Phi)\Phi = T. \tag{28}$$

Starobinsky model in Palatini formalism – Einstein frame

Let the new metric $\bar{g}_{\mu\nu}$ is given by $\bar{g}_{\mu\nu} = f'(\hat{R})g_{\mu\nu}$ and let $\hat{R}_{\mu\nu} = \bar{R}_{\mu\nu}, \bar{R} = \bar{g}^{\mu\nu}\bar{R}_{\mu\nu} = \Phi^{-1}\hat{R}$ and $\bar{g}_{\mu\nu}\bar{R} = \ g_{\mu\nu}\hat{R}$. Then equations (27a) and (27c) have the following form

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{U}(\Phi), \tag{29}$$

$$\Phi\bar{R} - (\Phi^2\,\bar{U}(\Phi))' = 0, \tag{30}$$

where we introduce $\bar{U}(\phi) = U(\phi)/\Phi^2$ and $\bar{T}_{\mu\nu} = \Phi^{-1}T_{\mu\nu}$. The structural equation is replaced by

$$\Phi\,\bar{U}'(\Phi) + \bar{T} = 0\,. \tag{31}$$

Palatini formalism in cosmology – Einstein frame

In this case, the action for the metric $\bar{g}_{\mu\nu}$ and scalar field Φ is given by

$$S(\bar{g}_{\mu\nu},\Phi)=\frac{1}{2}\int \mathrm{d}^4x\sqrt{-\bar{g}}\left(\bar{R}-\bar{U}(\Phi)\right)+S_m(\Phi^{-1}\bar{g}_{\mu\nu},\psi), \tag{32}$$

where

$$\bar{T}^{\mu\nu}=-\frac{2}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}_{\mu\nu}}S_m=(\bar{\rho}+\bar{p})\bar{u}^{\mu}\bar{u}^{\nu}+\bar{p}\bar{g}^{\mu\nu}=\Phi^{-3}T^{\mu\nu}\ , \tag{33}$$

and $\bar{u}^{\mu}=\Phi^{-\frac{1}{2}}u^{\mu}$, $\bar{\rho}=\Phi^{-2}\rho,\ \bar{p}=\Phi^{-2}p,\ \bar{T}_{\mu\nu}=\Phi^{-1}T_{\mu\nu},\ \bar{T}=\Phi^{-2}T$.

Starobinsky model in Palatini formalism – Einstein frame

In FRW case, we have the metric in the following form

$$d\bar{s}^2 = d\bar{t}^2 - \bar{a}^2(t)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{34}$$

where $d\bar{t} = \Phi(t)^{\frac{1}{2}} d\, t$ and new scale factor $\bar{a}(\bar{t}) = \Phi(\bar{t})^{\frac{1}{2}} a(\bar{t})$.
The cosmological equations (in the case of the barotropic matter) are given by

$$3\bar{H}^2 = \bar{\rho}_\Phi + \bar{\rho}_m, \quad 3\frac{\ddot{\bar{a}}}{\bar{a}} = \bar{\rho}_\Phi - \bar{\rho}_m(1+3w)$$

where $\bar{\rho}_\Phi = \frac{1}{2}\bar{U}(\Phi), \bar{\rho}_\mathrm{m} = \rho_0 \bar{a}^{-3(1+w)}\Phi^{\frac{1}{2}(3w-1)}$ and $w = \bar{p}_\mathrm{m}/\bar{\rho}_\mathrm{m}$. In this case, the conservation equations has the following form

$$\dot{\bar{\rho}}_\mathrm{m} + 3\bar{H}\bar{\rho}_\mathrm{m}(1+w) = -\dot{\bar{\rho}}_\Phi.$$

Starobinsky model in Palatini formalism – Einstein frame

Let us consider the Starobinsky model in FRW geometry in details. The potential $\bar{U}$ is described by the following formula

$$\bar{U}(\Phi) = 2\bar{\rho}_\Phi(\Phi) = \left(\frac{1}{4\gamma} + 2\lambda\right)\frac{1}{\Phi^2} - \frac{1}{2\gamma}\frac{1}{\Phi} + \frac{1}{4\gamma}. \tag{35}$$

Note that the function $\bar{\rho}_\Phi$ has the same shape like the Starobinsky potential. The function $\bar{\rho}_\Phi(\Phi)$ has the minimum for

$$\Phi = 1 + 8\gamma\lambda. \tag{36}$$

Starobinsky model in Palatini formalism – Einstein frame

The scalar field $\Phi(\bar{a})$ is given by

$$\Phi(\bar{a}) = 1 + 8\gamma\lambda + 2\gamma\rho_m - 6\gamma p_m. \tag{37}$$

Because $\bar{\rho}_m = \Phi^{-2}\rho_m$, $\bar{p}_m = \Phi^{-2}p_m$, from equation (37) we get

$$(2\gamma\bar{\rho}_m - 6\gamma\bar{p}_m)\Phi^2 - \Phi(\bar{a}) + 1 + 8\gamma\lambda = 0. \tag{38}$$

Because we assume the positive Φ, equation (38) gives the following formula for Φ in the case of the positive parameter γ

$$\Phi(\bar{a}) = \frac{1 + \sqrt{1 - 8\gamma(\bar{\rho}_m - 3\bar{p}_m)(1 + 8\gamma\lambda)}}{4\gamma(\bar{\rho}_m + 4\bar{p}_m)} \tag{39}$$

and the negative parameter γ

$$\Phi(\bar{a}) = \frac{1 - \sqrt{1 - 8\gamma(\bar{\rho}_m - 3\bar{p}_m)(1 + 8\gamma\lambda)}}{4\gamma(\bar{\rho}_m + 4\bar{p}_m)}. \tag{40}$$

Starobinsky model in Palatini formalism – Einstein frame

From formulas (39) and (40), we get the following condition for $\bar{\rho}_m + 4\bar{p}_m$ for the positive parameter γ

$$\bar{\rho}_m - 3\bar{p}_m < \frac{1}{8\gamma(1 + 8\gamma\lambda)} \tag{41}$$

and for the negative parameter γ

$$\bar{\rho}_m - 3\bar{p}_m > \frac{1}{8\gamma(1 + 8\gamma\lambda)}. \tag{42}$$

For $\gamma \approx 0$, the potential $\bar{U}$ can be approximated as $\bar{U} = -\bar{\rho}_m + \frac{1}{4\gamma}$. In this case the Friedmann equation can be written as

$$3\bar{H}^2 = \frac{\bar{\rho}_m}{2} + \frac{1}{8\gamma}. \tag{43}$$

In the case of $\bar{\rho}_m = 0$, $\bar{\rho}_\Phi$ is constant and the Friedmann equation has the following form

$$3\bar{H}^2 = \frac{1}{8\gamma}. \tag{44}$$

Starobinsky model in Palatini formalism – Inflation

Now we consider the function (39) with the positive parameter γ. In this model the inflation phenomenon appears when the the value of the parameter γ is close to zero and the matter $\bar{\rho}_m$ is negligible with comparison to $\bar{\rho}_\Phi$. In this case the approximate number of e-foldings is given by the following formula

$$N = H_{\text{init}}(\bar{t}_{\text{fin}} - \bar{t}_{\text{init}}) = \frac{\bar{t}_{\text{fin}} - \bar{t}_{\text{init}}}{\sqrt{24\gamma}}. \tag{45}$$

The number of e-folds N should be equal $50 \sim 60$ in the inflation epoch. In this model we obtain $N = 60$, when $\gamma = 1.16 \times 10^{-69}$ s^2 and the timescale of the inflation is equal 10^{-32} s.
The condition for appearing of the inflation requires the value of the parameter γ to be close to zero, hence the influence of the parameter λ for the evolution of the universe is negligible.

Starobinsky model in Palatini formalism – Inflation

In this model the inflation appears when matter $\bar{\rho}_{\mathrm{m}}$ is negligible with comparison to $\bar{\rho}_{\phi}$.
In statistical analysis the slow roll parameters are helpful in the estimation of model parameter in the inflation period. These parameters are defined as

$$\epsilon = -\frac{\dot{H}}{H^2} \text{ and } \eta = 2\epsilon - \frac{\dot{\epsilon}}{2H\epsilon}. \tag{46}$$

In our model the slow roll parameters have the following form

$$\epsilon = \frac{3}{2}\frac{\hat{R} - 4\Lambda(1+2\gamma\hat{R})^2}{\hat{R} + \frac{\gamma}{2}\hat{R}^2 - 3\Lambda(1+2\gamma\hat{R})^2}, \tag{47}$$

$$\eta = 5 + \frac{3}{2(\gamma\hat{R} - 1)} + \frac{\hat{R}(1+2\gamma\hat{R})}{6\Lambda(1+\gamma\hat{R})^2 - \hat{R}(2+\gamma\hat{R})}. \tag{48}$$

Starobinsky model in Palatini formalism – Inflation

From the Planck observations, we know a limit at a 2-σ level of the values of the scalar spectral index n_s and the tensor-to-scalar ratio r ($n_s = 0.9667 \pm 0.0040$ and $r < 0.113$). The relation between the scalar spectral index and the tensor-to-scalar ratio and the slow roll parameters are the following

$$n_s - 1 = -6\epsilon + 2\eta \quad \text{and} \quad r = 16\epsilon. \tag{49}$$

Because the slow roll parameters ϵ and η cannot be treated as constant parameters in our model, then we cannot use these parameters to find the restriction on the parameter γ from astronomical observations.

Starobinsky model in Palatini formalism – Inflation

For example, if we assume that $\frac{\Lambda}{3H_0^2} = 0.6911$, where $H_0 = 67.74 \frac{\mathrm{km}}{\mathrm{s\ Mpc}}$ then we get that $3.277 \times 10^{-6} \frac{\mathrm{s^2 Mpc^2}}{\mathrm{km^2}} < \gamma < 3.285 \times 10^{-6} \frac{\mathrm{s^2 Mpc^2}}{\mathrm{km^2}}$, $0 < \Omega_{\mathrm{m}} = \frac{\bar{\rho}_{\mathrm{m}}}{3\bar{H}^2} < 0.0047$ and $\Omega_{\Phi} = \frac{\bar{\rho}_{\Phi}}{3\bar{H}^2} \approx 0.50$. But this value of the parameter γ is too large in order to explain the present evolution of the Universe. In consequence, the slow roll parameters are useless in the estimation of the parameter γ.

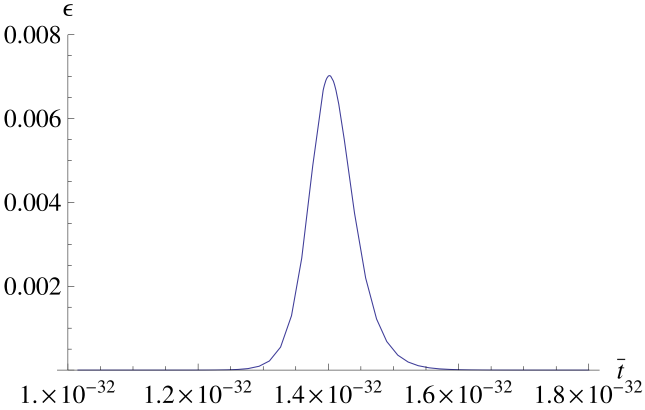

Figure: Diagram presents the evolution of ϵ with respect to the cosmological time $\bar{t}$. The time is expressed in seconds. The value of the parameter γ is assumed as $3.277 \times 10^{-6} \frac{s^2 Mpc^2}{km^2}$ and we assume that $\frac{\Lambda}{3H_0^2} = 0.6911$, where $H_0 = 67.74 \frac{km}{s\ Mpc}$.

Starobinsky model in Palatini formalism – Inflation

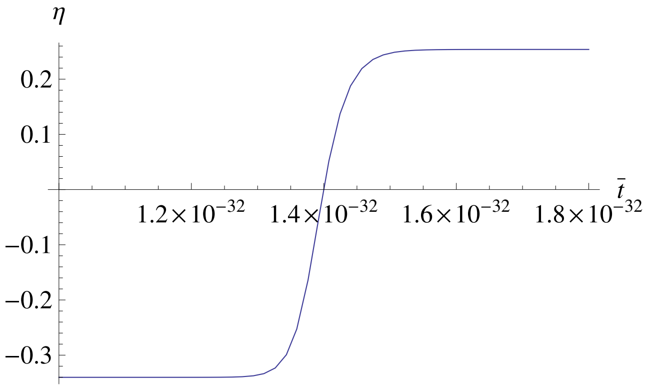

Figure: Diagram presents the evolution of η with respect to the cosmological time $\bar{t}$. The time is expressed in seconds. The value of the parameter γ is assumed as $3.277 \times 10^{-6} \frac{s^2 \mathrm{Mpc}^2}{\mathrm{km}^2}$ and we assume that $\frac{\Lambda}{3H_0^2} = 0.6911$, where $H_0 = 67.74 \frac{\mathrm{km}}{\mathrm{s\ Mpc}}$.

Starobinsky model in Palatini formalism – Inflation

The evolution of matter in the inflation period can be divided into four phases. The first phase is when matter is negligible and the density of ρ_{m} increases by the interaction with the potential ρ_{Φ}. The second phase is when the matter cannot be negligible and its density still increases. In this phase the injection of matter is the most effective. After achieving of the maximum of the density of ρ_{m} the third phase appears. In this phase matter still cannot be negligible but its density decreases. The last phase is when matter density decreases and is negligible. The maximum is achieved when

$$\hat{R} = \frac{1}{2\gamma}. \tag{50}$$

In the maximum, the value of $\bar{\rho}_{\mathrm{m}}$ is equal to $\frac{1}{8\gamma} - 4\Lambda$.

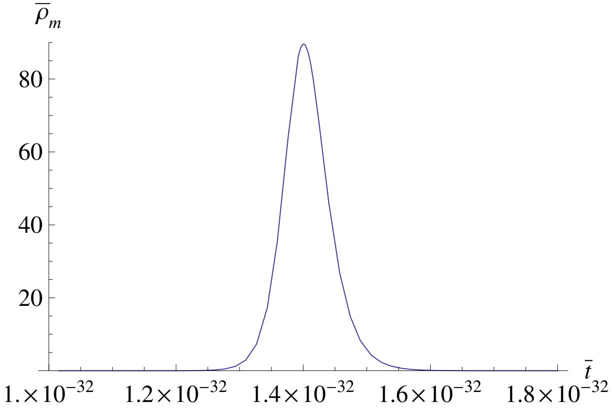

Figure: Diagram presents the evolution of $\bar{\rho}_{\text{m}}$ with respect to the cosmological time $\bar{t}$. The time is expressed in seconds and $\bar{\rho}_{\text{m}}$ is expressed in $\frac{\text{km}^2}{\text{s}^2\text{Mpc}^2}$. The value of γ parameter is assumed as $3.277 \times 10^{-6}\frac{\text{s}^2\text{Mpc}^2}{\text{km}^2}$ and we assume that $\frac{\Lambda}{3H_0^2} = 0.6911$, where $H_0 = 67.74\frac{\text{km}}{\text{s Mpc}}$.

We can find four phases. In the four phase, ρ_Φ is constant and is equal

$$\rho_\Phi = \frac{1 - 16\gamma\Lambda + \sqrt{1 - 32\gamma\Lambda}}{8\gamma} \tag{51}$$

and in the last phase when ρ_Φ is also constant

$$\rho_\Phi = \frac{1 - 16\gamma\Lambda - \sqrt{1 - 32\gamma\Lambda}}{8\gamma} \tag{52}$$

for $\delta = 0$. The difference between ρ_Φ in the first and in the last phase is equal

$$\Delta\rho_\Phi = \frac{\sqrt{1 - 32\gamma\Lambda}}{4\gamma} \approx \frac{1}{4\gamma}. \tag{53}$$

Starobinsky model in Palatini formalism – Inflation

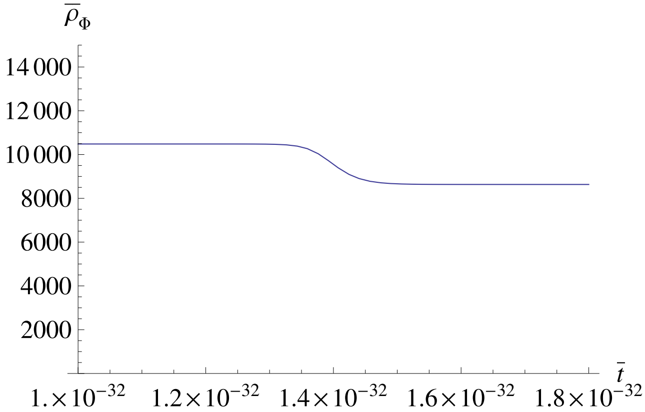

Figure: Diagram presents the evolution of $\bar{\rho}_\Phi$ with respect to the cosmological time $\bar{t}$. The time is expressed in seconds and $\bar{\rho}_\Phi$ is expressed in $\frac{\text{km}^2}{\text{s}^2\text{Mpc}^2}$. The value of γ parameter is assumed as $3.277 \times 10^{-6}\frac{\text{s}^2\text{Mpc}^2}{\text{km}^2}$ and we assume that $\frac{\Lambda}{3H_0^2} = 0.6911$, where $H_0 = 67.74\frac{\text{km}}{\text{s Mpc}}$.

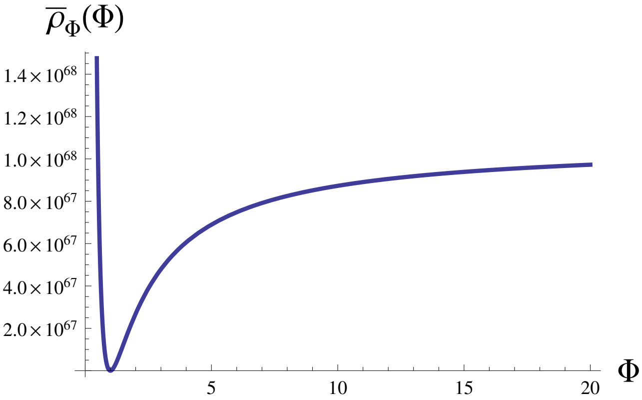

Figure: Illustration of the dependence $\bar{\rho}_{\Phi}$ of Φ. We assume that $\gamma = 1.16 \times 10^{-69}$ s^2. The units of $\bar{\rho}_{\Phi}$ are expressed in $\frac{\mathrm{km}^2}{\mathrm{s}^2\mathrm{Mpc}^2}$. Note that this potential has the same shape like the Starobinsky potential.

Starobinsky model in Palatini formalism – Inflation

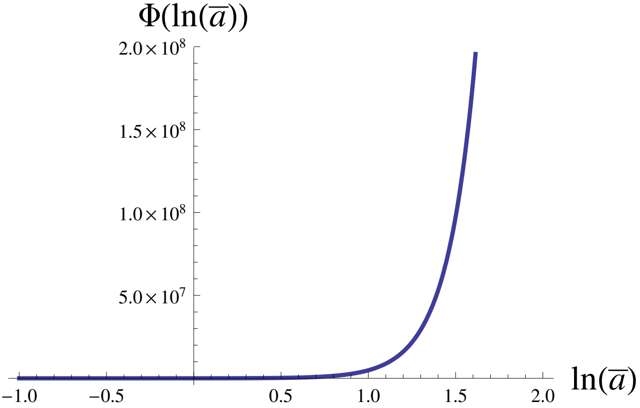

Figure: Illustration of the typical evolution of Φ with respect to $\ln(\bar{a})$ at the beginning of the inflation epoch. We assume that $\gamma = 1.16 \times 10^{-69}$ s^2 and $\bar{a}_0 = 1$ at the beginning of the inflation epoch.

Starobinsky model in Palatini formalism – Inflation

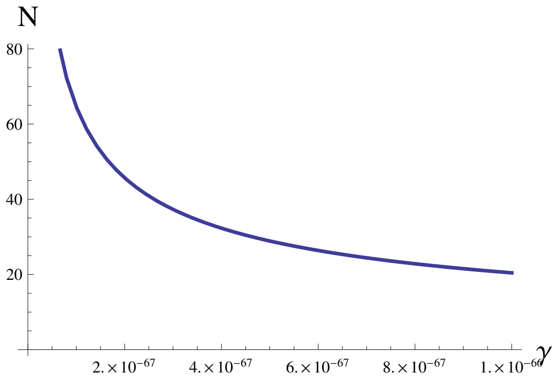

Figure: The diagram of the relation between γ and the approximate number of e-foldings $N = \bar{H}_{\mathrm{init}}(\bar{t}_{\mathrm{fin}} - \bar{t}_{\mathrm{init}})$ from $\bar{t}_{\mathrm{init}}$ to $\bar{t}_{\mathrm{fin}}$. We assume that $\bar{t}_{\mathrm{fin}} - \bar{t}_{\mathrm{init}} \approx 10^{-32}$ s. The units of the parameter γ are expressed in s^2. Note that the number of e-foldings grows when the parameter γ decreases and $N = 60$ when $\gamma = 1.16 \times 10^{-69}$ s^2.

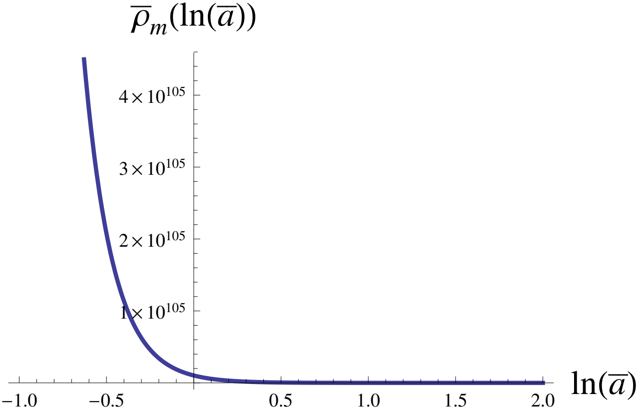

Figure: Illustration of the typical evolution of $\bar{\rho}_m$ with respect to $\ln(\bar{a})$ at the beginning of the inflation epoch. We assume that $\gamma = 1.16 \times 10^{-69}$ s^2 and $\bar{a}_0 = 1$ at the beginning of the inflation epoch. The units of $\bar{\rho}_m$ are expressed in $\frac{\mathrm{km}^2}{\mathrm{s}^2\mathrm{Mpc}^2}$.

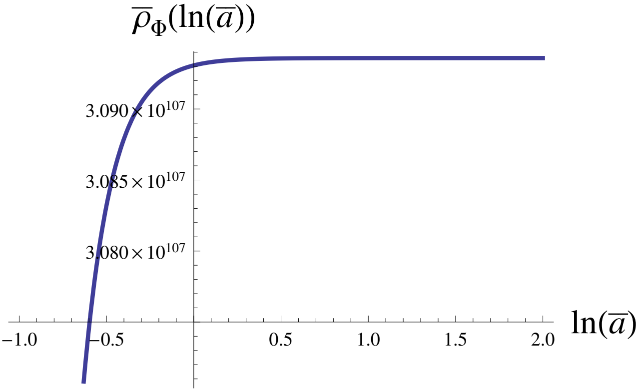

Figure: Illustration of the typical evolution of $\bar{\rho}_\phi$ with respect to $\ln(\bar{a})$ at the beginning of the inflation epoch. We assume that $\gamma = 1.16 \times 10^{-69}$ s^2 and $\bar{a}_0 = 1$ at the beginning of the inflation epoch. The units of $\bar{\rho}_\Phi$ are expressed in $\frac{\mathrm{km}^2}{\mathrm{s}^2\mathrm{Mpc}^2}$. Note that during the inflation $\bar{\rho}_\phi \approx$ const.

Starobinsky model in Palatini formalism – Inflation

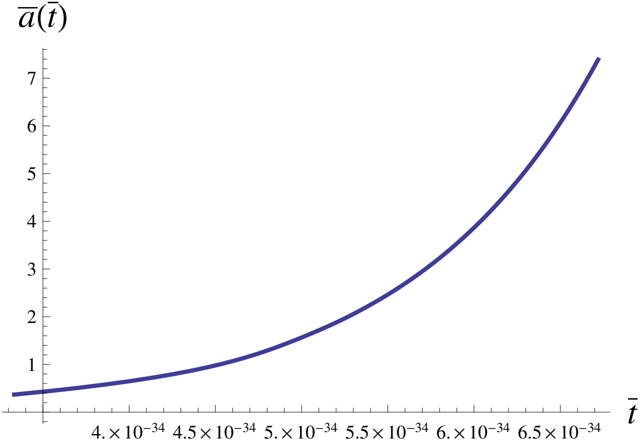

Figure: Illustration of the typical evolution of $\bar{a}$ with respect to $\bar{t}$ at the beginning of the inflation epoch. We assume that $\gamma = 1.16 \times 10^{-69}$ s^2 and $\bar{a}_0 = 1$ at the beginning of the inflation epoch. The time $\bar{t}$ is expressed in seconds.

Starobinsky model in Palatini formalism – Inflation

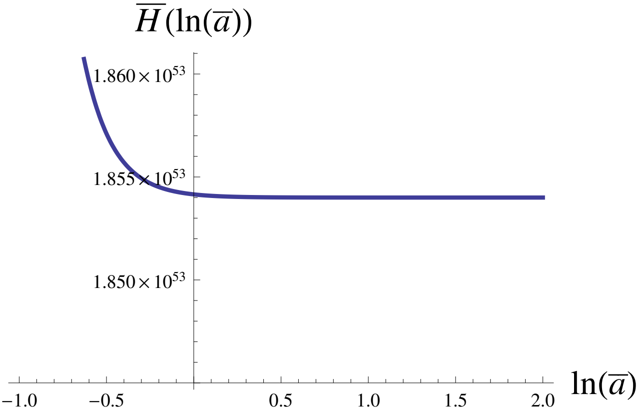

Figure: Illustration of the typical evolution of $\bar{H}$ with respect to $\ln(\bar{a})$ at the beginning of the inflation epoch. We assume that $\gamma = 1.16 \times 10^{-69}$ s^2 and $\bar{a}_0 = 1$ at the beginning of the inflation epoch. The units of $\bar{H}$ are expressed in $\frac{\text{km}}{\text{s Mpc}}$. Note that for the late time, $\bar{H}$ can be treated as a constant.

Starobinsky model in Palatini formalism in Jordan frame – dynamical analysis

Following their classification the type of singularity:

- Type 0: 'Big crunch'. In this type, the scale factor a is vanishing and blow up of the Hubble parameter H, energy density ρ and pressure p.
- Type I: 'Big rip'. In this type, the scale factor a, energy density ρ and pressure p are blown up.
- Type II: 'Sudden'. The scale factor a, energy density ρ and Hubble parameter H are finite and $\dot{H}$ and the pressure p are divergent.
- Type III: 'Big freeze'. The scale factor a is finite and the Hubble parameter H, energy density ρ and pressure p are blown up or divergent.
- Type IV. The scale factor a, Hubble parameter H, energy density ρ, pressure p and $\dot{H}$ are finite but higher derivatives of the scale factor a diverge.
- Type V. The scale factor a is finite but the energy density ρ and pressure p vanish.

Starobinsky model in Palatini formalism in Jordan frame – dynamical analysis

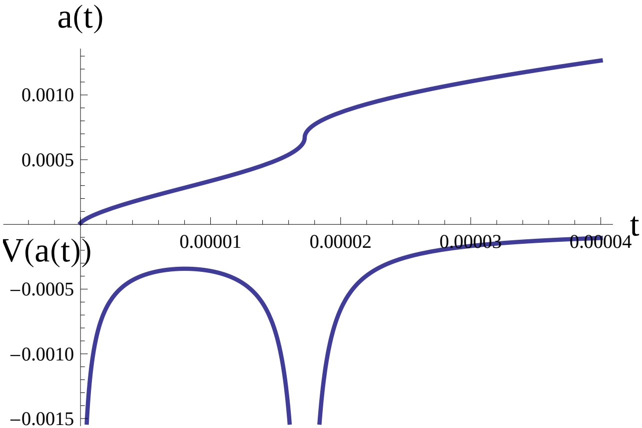

Figure: Illustration of sewn freeze singularity, when the potential $V(a)$ has a pole.

Starobinsky model in Palatini formalism in Jordan frame – dynamical analysis

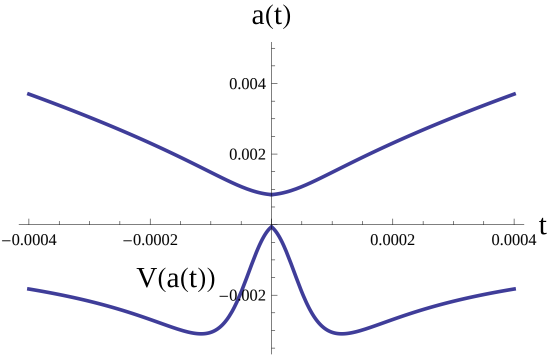

Figure: Illustration of a sewn sudden singularity. The model with negative Ω_γ has a mirror symmetry with respect to the cosmological time. Note that the spike on the diagram shows discontinuity of the function $\frac{\partial V}{\partial a}$. Note the existence of a bounce at $t = 0$.

Starobinsky model in Palatini formalism in Jordan frame – dynamical analysis

In our model, one finds two types of singularities, which are a consequence of the Palatini formalism: the freeze and sudden singularity. The freeze singularity appears when the multiplicative expression $\frac{b}{b+d/2}$, in the Friedmann equation (15), is equal the infinity. So we get a condition for the freeze singularity: $2b+d=0$ which produces a pole in the potential function. It appears that the sudden singularity appears in our model when the multiplicative expression $\frac{b}{b+d/2}$ vanishes. This condition is equivalent to the case $b=0$.

The freeze singularity in our model is a solution of the algebraic equation

$$2b+d=0 \Longrightarrow f(K,\Omega_{\Lambda,0},\Omega_{\gamma})=0 \tag{54}$$

or

$$-3K-\frac{K}{3\Omega_{\gamma}(\Omega_{\mathrm{m}}+\Omega_{\Lambda,0})\Omega_{\Lambda,0}}+1=0, \tag{55}$$

where $K\in[0,\ 3)$.

Starobinsky model in Palatini formalism in Jordan frame – dynamical analysis

The solution of the previous equation is

$$K_{\mathrm{freeze}} = \frac{1}{3 + \frac{1}{3\Omega_\gamma(\Omega_{\mathrm{m}}+\Omega_{\Lambda,0})\Omega_{\Lambda,0}}}. \tag{56}$$

From equation (56), we can find an expression for a value of the scale factor for the freeze singularity

$$a_{\mathrm{freeze}} = \left(\frac{1-\Omega_{\Lambda,0}}{8\Omega_{\Lambda,0} + \frac{1}{\Omega_\gamma(\Omega_{\mathrm{m}}+\Omega_{\Lambda,0})}}\right)^{\frac{1}{3}}. \tag{57}$$

Starobinsky model in Palatini formalism in Jordan frame – dynamical analysis

The sudden singularity appears when $b = 0$. This provides the following algebraic equation

$$1 + 2\Omega_\gamma(\Omega_{\mathrm{m},0}a^{-3} + \Omega_{\Lambda,0})(K+1) = 0. \tag{58}$$

The above equation can be rewritten as

$$1 + 2\Omega_\gamma(\Omega_{\mathrm{m},0}a^{-3} + 4\Omega_{\Lambda,0}) = 0. \tag{59}$$

From the equation (59), we have the formula for the scale factor for sudden singularity

$$a_{\mathrm{sudden}} = \left(-\frac{2\Omega_{\mathrm{m},0}}{\frac{1}{\Omega_\gamma} + 8\Omega_{\Lambda,0}}\right)^{1/3}. \tag{60}$$

which, in fact, becomes a (degenerate) critical point and a bounce at the same time.

Starobinsky model in Palatini formalism in Jordan frame – dynamical analysis

There is a class of cosmological models, which dynamics can be reduced to the dynamical system of the Newtonian type. Let consider a homogeneous and isotropic universe with a spatially flat space-time metric of the form

$$ds^2 = dt^2 - a^2(t)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{61}$$

where $a(t)$ is the scale factor and t is the cosmological time.
Let us consider the energy-momentum tensor T^μ_ν for the perfect fluid with energy density $\rho(t)$ and pressure $p(t)$ as a source of gravity. In this case the Einstein equations assumes the form of Friedmann equations

$$\rho = 3H^2 = \frac{3\dot{a}^2}{a^2}, \tag{62}$$

$$p = -\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}, \tag{63}$$

where dot denotes differentiation with respect to the cosmic time t, $H \equiv \frac{\dot{a}}{a}$ is the Hubble function.

Starobinsky model in Palatini formalism in Jordan frame – dynamical analysis

We assume that $\rho(t) = \rho(a(t))$ as well as $p(t) = p(a(t))$, i.e. that both energy density as well as pressure depends on the cosmic time through the scale factor $a(t)$. The conservation condition $T^{\mu\nu}_{;\mu} = 0$ reduces to

$$\dot{\rho} = -3H(\rho + p). \tag{64}$$

It would be convenient to rewrite (62) in an equivalent form

$$\dot{a}^2 = -2V(a), \tag{65}$$

where

$$V(a) = -\frac{\rho(a)a^2}{6}. \tag{66}$$

Starobinsky model in Palatini formalism in Jordan frame – dynamical analysis

In (66) $\rho(a)$ plays the model role of effective energy density. For example for the standard cosmological model (14)

$$V = -\frac{\rho_{\text{eff}} a^2}{6} = -\frac{a^2}{6}\left(\rho_{m,0} a^{-3} + \rho_{\Lambda,0}\right), \tag{67}$$

where $\rho_{\text{eff}} = \rho_{\text{m}} + \Lambda$ and $\rho_{\text{m}} = \rho_{\text{m},0} a^{-3}$. Equation (63) is equivalent to

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p), \tag{68}$$

which is called acceleration equation. It is easily to check that

$$\ddot{a} = -\frac{\partial V}{\partial a}, \tag{69}$$

where $V(a)$ is given by (66) provided that conservation equation (64) is fulfilled.

Starobinsky model in Palatini formalism in Jordan frame – dynamical analysis

Due to equation (69) the evolution of a universe can be interpreted as a motion of a fictitious particle of unit mass in the potential $V(a)$. Here $a(t)$ plays the role of a positional variable. Equation of motion (69) assumes the form analogous to the Newtonian equation of motion.
If we know the form of effective energy density then we can construct the form of potential $V(a)$, which determine the whole dynamics in the phase space $(a, \dot{a})$. In this space the Friedmann equation (65) plays the role of a first integral and determines the phase space curves representing the evolutionary paths of the cosmological models. The diagram of potential $V(a)$ contains all information needed to construction of a phase space portrait. In this case the phase space is two-dimensional

$$\left\{(a, \dot{a}) \colon \frac{\dot{a}^2}{2} + V(a) = -\frac{k}{2}\right\}. \tag{70}$$

Starobinsky model in Palatini formalism in Jordan frame – dynamical analysis

In a general case of arbitrary potential, a dynamical system which describes the evolution of a universe takes the form

$$\dot{a} = x, \tag{71}$$

$$\dot{x} = -\frac{\partial V(a)}{\partial a}. \tag{72}$$

We shall study the system above using theory of piece-wise smooth dynamical systems. Therefore it is assumed that the potential function, except some isolated (singular) points, belongs to the class $C^2(\mathbb{R}_+)$.

Starobinsky model in Palatini formalism in Jordan frame – dynamical analysis

The lines $\frac{x^2}{2} + V(a) = -\frac{k}{2}$ represent possible evolutions of the universe for different initial conditions. The equations (71) and (72) can be rewritten in dimensionless variables if we replace effective energy density ρ_{eff} by density parameter

$$\Omega_{\text{eff}} = \frac{\rho_{\text{eff}}}{3H_0^2}, \tag{73}$$

then

$$\frac{1}{H_0^2}\frac{\dot{a}^2}{2} = -\frac{\Omega_{\text{eff}}a^2}{2}, \tag{74}$$

$$\frac{d^2a}{d\tau^2} = -\frac{\partial\tilde{V}}{\partial a}, \tag{75}$$

where $t \to \tau = |H_0|t$ and

$$\tilde{V}(a) = -\frac{\Omega_{\text{eff}}a^2}{2}. \tag{76}$$

Starobinsky model in Palatini formalism in Jordan frame – dynamical analysis

Any cosmological model can be identified by its form of the potential function $V(a)$ depending on the scale factor a. From the Newtonian form of the dynamical system (71)-(72) one can see that all critical points correspond to vanishing of r.h.s of the dynamical system $\left(x_0 = 0,\ \frac{\partial V(a)}{\partial a}|_{a=a_0}\right)$. Therefore all critical points are localized on the x-axis, i.e. they represent a static universe.

Starobinsky model in Palatini formalism in Jordan frame – dynamical analysis

Because of the Newtonian form of the dynamical system the character of critical points is determined from the characteristic equation of the form

$$a^2 + \det A\big|_{x_0=0, \frac{\partial V(a)}{\partial a}|_{a_0}=0} = 0, \tag{77}$$

where $\det A$ is determinant of linearization matrix calculated at the critical points, i.e.

$$\det A = \frac{\partial^2 V(a)}{\partial a^2}\Big|_{a_0, \frac{\partial V(a)}{\partial a}|_{a_0}=0}. \tag{78}$$

From equation (77) and (78) one can conclude that only admissible critical points are the saddle type if $\frac{\partial^2 V(a)}{\partial a^2}|_{a=a_0} < 0$ or centres type if $\frac{\partial^2 V(a)}{\partial a^2}|_{a=a_0} > 0$.

Starobinsky model in Palatini formalism in Jordan frame – dynamical analysis

Let $V = -\frac{a^2}{2}\left(\Omega_\gamma(\Omega_{\mathrm{m}} + \Omega_\Lambda)^2 \frac{(K-3)(K+1)}{2b} + (\Omega_{\mathrm{m}} + \Omega_\Lambda) + \Omega_k\right)$. We can rewrite dynamical system (71)-(72) as

$$a' = x, \tag{79}$$

$$x' = -\frac{\partial V(a)}{\partial a}, \tag{80}$$

where $' \equiv \frac{d}{d\sigma} = \frac{b+\frac{d}{2}}{b}\frac{d}{d\tau}$ is a new parametrization of time.
We can treated the dynamical system (79)-(80) as a sewn dynamical system. In this case, we divide the phase portrait into two parts: the first part is for $a < a_{\mathrm{sing}}$ and the second part is for $a > a_{\mathrm{sing}}$. Both parts are glued along the singularity.

Starobinsky model in Palatini formalism in Jordan frame – dynamical analysis

For $a < a_{\text{sing}}$, dynamical system (79)-(80) can be rewritten to the corresponding form

$$a' = x, \tag{81}$$

$$x' = -\frac{\partial V_1(a)}{\partial a}, \tag{82}$$

where $V_1 = V(-\eta(a - a_s) + 1)$ and $\eta(a)$ notes the Heaviside function. For $a > a_{\text{sing}}$, in the analogous way, we get the following equations

$$a' = x, \tag{83}$$

$$x' = -\frac{\partial V_2(a)}{\partial a}, \tag{84}$$

where $V_2 = V\eta(a - a_s)$.

Starobinsky model in Palatini formalism in Jordan frame – dynamical analysis

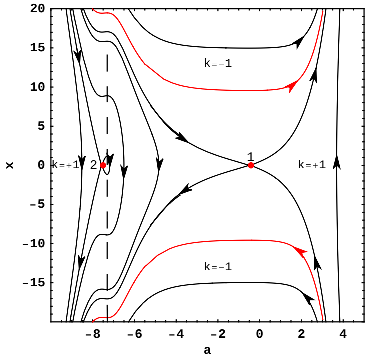

Figure: The Figure represents the phase portrait of the system (79-80) for positive Ω_γ. The scale factor a is in the logarithmic scale. The red trajectories represent the spatially flat universe. The dashed line $2b + d = 0$ corresponds to the freeze singularity. The critical points (1) and (2) present two static Einstein universes.

Starobinsky model in Palatini formalism in Jordan frame – dynamical analysis

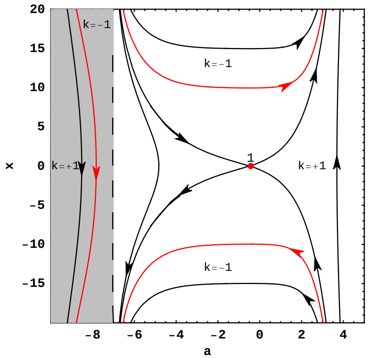

Figure: The phase portrait of the system (79-80) for negative Ω_γ. The scale factor a is in logarithmic scale. The red trajectories represent a spatially flat universe. The dashed line $b = 0$ corresponds to the sudden singularity. The shaded region represents trajectories with $b < 0$.

Starobinsky model in Palatini formalism in Einstein frame – dynamical analysis

If we consider dynamics in the Jordan frame then one can used a formula for H^2 for reducing dynamics to the dynamical system of the Newtonian type which possesses the first integral $\frac{1}{2}\left(\frac{da}{dt}\right)^2 + V(a) = 0$, where $V(a) = -\frac{1}{2}H^2a^2$. In this representation of dynamics, singularities for the finite value of the scale factor $a = a_s$ are poles of $V(a)$ potential or their derivatives. The generic feature of the formulation of dynamics is the appearance of the freeze or typical sudden type of singularity in the past. At the freeze singularity point while the scale factor is finite, its second derivative with respect to the time blows up, i.e $\frac{d^2a}{dt^2} = \pm\infty$. In general, all singularities can be detected from the diagram of the potential function.

Starobinsky model in Palatini formalism in Einstein frame – dynamical analysis

If we consider dynamics in the Einstein frame there are no such singularities. The big bang singularity present in the ΛCDM model is replaced by the generalized sudden singularity of the finite scale factor. Beyond this singularity, the phase portrait is equivalent to the ΛCDM model.

Two dynamical systems in the phase space are equivalent if there is a homeomorphism transforming all trajectories with the preserving of the direction of time measured along the trajectories. The comparison of dynamics in both the Jordan and Einstein frame explicitly shows that corresponding dynamical systems are not topologically equivalent. In consequence, the physics in both frames is different.

Starobinsky model in Palatini formalism in Einstein frame – dynamical analysis

The cosmological equation for the Starobinsky–Palatini model in the Einstein frame can be rewritten to the form of the dynamical system with the Hubble parameter $\bar{H}(\bar{t})$ and the Ricci scalar $\hat{R}(\bar{t})$ as variables

$$\dot{\bar{H}}(\bar{t}) = \frac{1}{6\ (1+2\gamma\hat{R}(\bar{t}))^2} \left(6\Lambda - 6\bar{H}(\bar{t})^2(1+2\gamma\hat{R}(\bar{t}))^2 + \hat{R}(\bar{t})(-1+24\gamma\Lambda+\gamma(1+24\gamma\Lambda)\hat{R}(\bar{t})\right), \tag{85}$$

$$\dot{\hat{R}}(\bar{t}) = -\frac{3}{(-1+\gamma\hat{R}(\bar{t}))}\ \bar{H}(\bar{t})(1+2\gamma\hat{R}(\bar{t})) \left(4\Lambda + \hat{R}(\bar{t})\left(-1+16\gamma\Lambda+16\gamma^2\Lambda\hat{R}(\bar{t})\right)\right), \tag{86}$$

where a dot denotes the differentiation with respect to the time $\bar{t}$.

Starobinsky model in Palatini formalism in Einstein frame – dynamical analysis

For the equations (85)–(86) and (90)–(91), we can find the first integrals. In the case of equations (85)–(86), the first integral has the following form

$$\bar{H}(\bar{t})^2 + \Lambda - \frac{\hat{R}(\bar{t})(2+\gamma\hat{R}(\bar{t}))}{6(1+2\gamma\hat{R}(\bar{t}))^2} + \frac{k}{2\bar{a}^2} = 0. \tag{87}$$

Starobinsky model in Palatini formalism in Einstein frame – dynamical analysis

In this case the scale factor $\bar{a}$ is given in the following form

$$\bar{a} = \sqrt{\frac{C_0(1+2\gamma\hat{R}(\bar{t}))}{2e^{-\frac{\arctan\left(\frac{-1+16\gamma\Lambda+32\gamma^2\Lambda\hat{R}(\bar{t})}{\sqrt{-1+32\gamma\Lambda}}\right)}{3\sqrt{-1+32\gamma\Lambda}}}\sqrt{4\Lambda+\hat{R}(\bar{t})\left(-1+16\gamma\Lambda+16\gamma^2\Lambda\hat{R}(\bar{t})\right)}}}, \tag{88}$$

where $C_0 = \frac{\bar{a}_0^2 e^{-\frac{\arctan\left(\frac{-1+16\gamma\Lambda+32\gamma^2\Lambda\hat{R}(\bar{t}_0)}{\sqrt{-1+32\gamma\Lambda}}\right)}{3\sqrt{-1+32\gamma\Lambda}}}\sqrt{4\Lambda+\hat{R}(\bar{t}_0)\left(-1+16\gamma\Lambda+16\gamma^2\Lambda\hat{R}(\bar{t}_0)\right)}}{(1+2\gamma\hat{R}(\bar{t}_0))}$ with $\bar{a}_0$ as the present value of the scale factor.

Starobinsky model in Palatini formalism in Einstein frame – dynamical analysis

We get the first integral in the following form

$$\bar{H}(\bar{t})^2 + \Lambda - \frac{\hat{R}(\bar{t})(2+\gamma\hat{R}(\bar{t}))}{6(1+2\gamma\hat{R}(\bar{t}))^2} + k\frac{e^{-\frac{\arctan\left(\frac{-1+16\gamma\Lambda+32\gamma^2\Lambda\hat{R}(\bar{t})}{\sqrt{-1+32\gamma\Lambda}}\right)}{3\sqrt{-1+32\gamma\Lambda}}}\sqrt{4\Lambda+\hat{R}(\bar{t})\left(-1+16\gamma\Lambda+16\gamma^2\Lambda\hat{R}(\bar{t})\right)}}{C_0(1+2\gamma\hat{R}(\bar{t}))} = 0. \tag{89}$$

Starobinsky model in Palatini formalism in Einstein frame – dynamical analysis

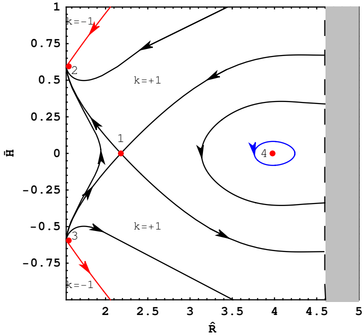

Figure: The phase portrait of system (85)-(86). There are four critical points: point 1 represents the Einstein universe, point 2 represents the stable de Sitter universe, point 3 represents the unstable de Sitter universe and point 4 represents the Einstein universe. The value of the parameter γ is chosen as $10^{-6}\frac{\mathrm{s^2 Mpc^2}}{\mathrm{km^2}}$.

Starobinsky model in Palatini formalism in Einstein frame – dynamical analysis

For comparison of the dynamical system in the both frames, we obtain dynamical system for the Starobinsky–Palatini model in the Jordan frame in the variables $H(t)$ and $\hat{R}(t)$

$$\dot{H}(t) = -\frac{1}{6}\left[6\left(2\Lambda + H(t)^2\right) + \hat{R}(t) + \frac{18(1+8\gamma\Lambda)\left(\Lambda - H(t)^2\right)}{-1-12\gamma\Lambda+\gamma\hat{R}(t)} - \frac{18(1+8\gamma\Lambda)H(t)^2}{1+2\gamma\hat{R}(t)}\right], \quad (90)$$

$$\dot{\hat{R}}(t) = -3H(t)(\hat{R}(t) - 4\Lambda), \quad (91)$$

where a dot means the differentiation with respect to time t. This phase portrait represents all evolutionary paths of the system in the Jordan frame without adopting the time reparameterization. Along the trajectories is measured original cosmological time t. The system (90)-(91) constitutes a two-dimensional autonomous dynamical system.

Starobinsky model in Palatini formalism in Einstein frame – dynamical analysis

For a deeper analysis of the behavior of the trajectories of system (90)–(91) in the infinity, we introduce variables $\hat{R}$ and $W = \frac{H}{\sqrt{1+H}}$ and rewrite equations (90)–(91) in these variables. Then we get the following dynamical system

$$\dot{W}(t) = \frac{\dot{H}(t)}{(1+H(t)^2)^{3/2}} = -\frac{1}{6}\left[6\left(2\Lambda + \frac{W(t)^2}{1-W(t)^2}\right) + \hat{R}(t) + \frac{18(1+8\gamma\Lambda)\left(\Lambda - \frac{W(t)^2}{1-W(t)^2}\right)}{-1-12\gamma\Lambda+\gamma\hat{R}(t)} - \frac{18(1+8\gamma\Lambda)\frac{W(t)^2}{1-W(t)^2}}{1+2\gamma\hat{R}(t)}\right], \quad (92)$$

$$\dot{\hat{R}}(t) = -3\frac{W(t)}{\sqrt{1-W(t)^2}}(\hat{R}(t) - 4\Lambda). \quad (93)$$

Starobinsky model in Palatini formalism in Einstein frame – dynamical analysis

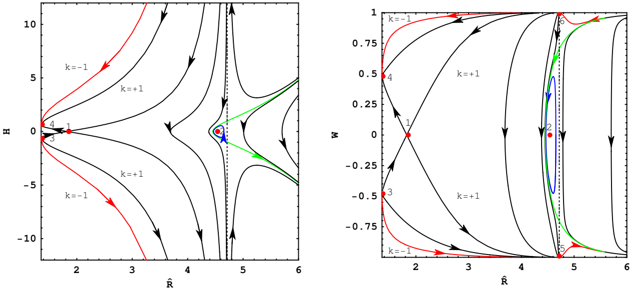

Figure: The left panel is the phase portrait of system (90)-(91) and the right one is the phase portrait of system (92)-(93). There are four critical points in the both systems: point 1 and 2 represent the Einstein universe, point 3 represents the unstable de Sitter universe, point 4 represents the stable de Sitter universe. For illustration the value of the parameter γ is chosen as $10^{-6}\,\frac{s^2 Mpc^2}{km^2}$.

Starobinsky model in Palatini formalism in Einstein frame – dynamical analysis

Equations (90)–(91) have the following the first integral given by

$$H(t)^2 - \frac{(1+2\gamma\hat{R}(t))^2\left(-3\Lambda + \hat{R}(t) - \frac{k(-4\Lambda+\hat{R}(t))^{2/3}}{C_0} + \frac{\gamma(12\Lambda-3\hat{R}(t))\hat{R}(t)}{2(1+2\gamma\hat{R}(t))}\right)}{(1+2\gamma\hat{R}(t)-3\gamma(-4\Lambda+\hat{R}(t)))^2} = 0, \quad (94)$$

where $C_0 = a_0^2(-4\Lambda + \hat{R}(t_0))^{2/3}$. Here, a_0 is the present value of the scale factor.

Statistical analysis of Starobinsky model in Palatini formalism in Jordan frame

In statistical analysis, in the case of the Starobinsky model in the Palatini formalism in Jordan frame, we used the following astronomical observations: observations of 580 supernovae of type Ia, BAO, measurements of $H(z)$ for galaxies, Alcock-Paczyński test, measurements of CMB and lensing by Planck and low ℓ by WMAP.

Statistical analysis of Starobinsky model in Palatini formalism in Jordan frame

The likelihood function for observations of supernovae of type Ia is given by the following expression

$$\ln L_{\mathrm{SNIa}} = -\frac{1}{2}[A - B^2/C + \ln(C/(2\pi))], \tag{95}$$

where $A = (\mu^{\mathrm{obs}} - \mu^{\mathrm{th}})\mathbb{C}^{-1}(\mu^{\mathrm{obs}} - \mu^{\mathrm{th}})$, $B = \mathbb{C}^{-1}(\mu^{\mathrm{obs}} - \mu^{\mathrm{th}})$, $C = \mathrm{Tr}\mathbb{C}^{-1}$ and $\mathbb{C}$ is a covariance matrix for observations of supernovae of type Ia. The distance modulus is defined by the formula $\mu^{\mathrm{obs}} = m - M$ (where m is the apparent magnitude and M is the absolute magnitude of observations of supernovae of type Ia) and $\mu^{\mathrm{th}} = 5\log_{10} D_L + 25$ (where the luminosity distance is $D_L = c(1+z)\int_0^z \frac{dz'}{H(z)}$).

Statistical analysis of Starobinsky model in Palatini formalism in Jordan frame

BAO observations such as Sloan Digital Sky Survey Release 7 (SDSS DR7) dataset at $z = 0.275$, 6dF Galaxy Redshift Survey measurements at redshift $z = 0.1$, and WiggleZ measurements at redshift $z = 0.44, 0.60, 0.73$ have the following likelihood function

$$\ln L_{\mathrm{BAO}} = -\frac{1}{2}\left(\mathbf{d}^{\mathrm{obs}} - \frac{r_s(z_d)}{D_V(\mathbf{z})}\right)\mathbb{C}^{-1}\left(\mathbf{d}^{\mathrm{obs}} - \frac{r_s(z_d)}{D_V(\mathbf{z})}\right), \tag{96}$$

where $r_s(z_d)$ is the sound horizon at the drag epoch.

Statistical analysis of Starobinsky model in Palatini formalism in Jordan frame

For the Alcock-Paczynski test we used the following expression for the likelihood function

$$\ln L_{AP} = -\frac{1}{2}\sum_i \frac{\left(AP^{th}(z_i) - AP^{obs}(z_i)\right)^2}{\sigma^2}. \tag{97}$$

where $AP(z)^{\mathrm{th}} \equiv \frac{H(z)}{z}\int_0^z \frac{dz'}{H(z')}$ and $AP(z_i)^{\mathrm{obs}}$ are observational data.

Statistical analysis of Starobinsky model in Palatini formalism in Jordan frame

The likelihood function for measurements of the Hubble parameter $H(z)$ of galaxies from is given by the expression

$$\ln L_{H(z)} = -\frac{1}{2}\sum_{i=1}^{N}\left(\frac{H(z_i)^{\mathrm{obs}} - H(z_i)^{\mathrm{th}}}{\sigma_i}\right)^2. \tag{98}$$

Statistical analysis of Starobinsky model in Palatini formalism in Jordan frame

We use the likelihood function for observations of CMB and lensing by Planck, and low-ℓ polarization from the WMAP (WP) in the following form

$$\ln L_{\text{CMB+lensing}} = -\frac{1}{2}(\mathbf{x}^{\text{th}} - \mathbf{x}^{\text{obs}})\mathbb{C}^{-1}(\mathbf{x}^{\text{th}} - \mathbf{x}^{\text{obs}}), \tag{99}$$

where $\mathbb{C}$ is the covariance matrix with the errors, $\mathbf{x}$ is a vector of the acoustic scale l_A, the shift parameter R and $\Omega_b h^2$ where

$$l_A = \frac{\pi}{r_s(z^*)} c \int_0^{z^*} \frac{dz'}{H(z')} \tag{100}$$

$$R = \sqrt{\Omega_{\text{m},0} H_0^2} \int_0^{z^*} \frac{dz'}{H(z')}, \tag{101}$$

where z^* is the redshift of the epoch of the recombination.

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The total likelihood function is expressed in the following form

$$L_{\text{tot}} = L_{\text{SNIa}} L_{\text{BAO}} L_{\text{AP}} L_{H(z)} L_{\text{CMB+lensing}}. \tag{102}$$

In estimation of model parameters, we use our own code CosmoDarkBox. The Metropolis-Hastings algorithm is used in this code.

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Table: The best fit and errors for the estimated model for the positive Ω_γ with $\Omega_{\mathrm{m},0}$ from the interval $(0.27, 0.33)$, Ω_γ from the interval $(0.0, 2.6 \times 10^{-9})$ and H_0 from the interval (66.0 (km/(s Mpc)), 70.0 (km/(s Mpc))). $\Omega_{\mathrm{b},0}$ is assumed as 0.048468. The redshift of matter-radiation equality is assumed as 3395. H_0, in the table, is expressed in km/(s Mpc). The value of reduced χ^2 of the best fit of our model is equal 0.187066 (for the ΛCDM model 0.186814).

parameter	best fit	68% CL	95% CL
H_0	68.10	$+1.07$ -1.24	$+1.55$ -1.82
$\Omega_{\mathrm{m},0}$	0.3011	$+0.0145$ -0.0138	$+0.0217$ -0.0201
Ω_γ	9.70×10^{-11}	$+1.3480 \times 10^{-9}$ -9.70×10^{-11}	$+2.2143 \times 10^{-9}$ -9.70×10^{-11}

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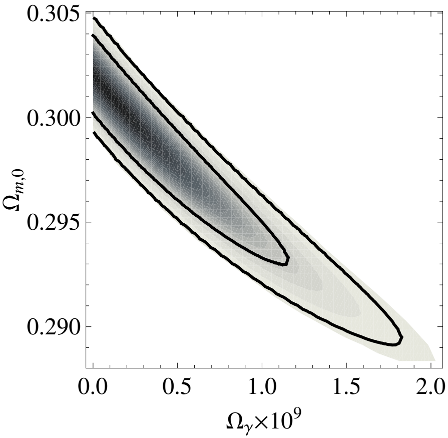

Figure: The intersection of the likelihood functions of two model parameters (Ω_γ, $\Omega_{m,0}$) with the marked 68% and 95% confidence levels.

Statistical analysis of Starobinsky model in Palatini formalism in Jordan frame

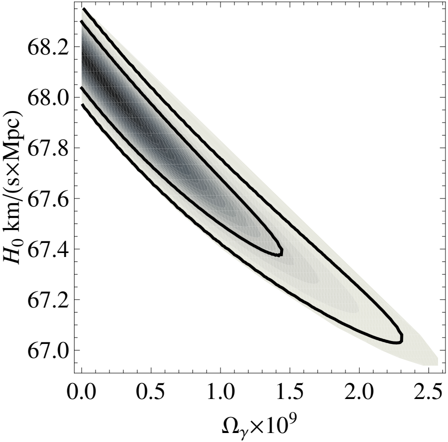

Figure: The intersection of the likelihood functions of two model parameters (Ω_γ, H_0) with the marked 68% and 95% confidence levels.

Summary

Detailed conclusions coming from our analysis are the following:

- We show that the interaction between the sectors of matter and the decaying vacuum appears naturally if we consider model formulation in the Einstein frame. For model formulated in the Jordan frame this interaction is absent.
- The inflation appears in our model formulated in the Einstein frame, when the parameter γ is close to zero and the density of matter is negligible in comparison to $\bar{\rho}_\Phi$.
- While the freeze double singularities appear in our model in the Jordan frame there is no such singularities in the dynamics of the model in the Einstein frame.

Summary

- In the context of the Starobinsky model in the Palatini formalism in the Jordan frame we found a new type of double singularity beyond the well-known classification of isolated singularities.
- The phase portrait for the Starobinsky model in the Palatini formalism in the Jordan frame with a positive value of γ is equivalent to the phase portrait of the ΛCDM model. There is only a quantitative difference related to the presence of the non-isolated freeze singularity.
- For the Starobinsky–Palatini model in the Einstein frame for the positive parameter γ, a sewn freeze singularity is replaced by a generalized sudden singularity. In consequence, this model is not equivalent to the phase portrait of the ΛCDM model.

Summary

- For the Starobinsky model in the Palatini formalism in the Einstein frame, we get the inflation for the early Universe.
- Given two representations of our model in the Einstein and Jordan frames, we found that its dynamics is simpler in the Einstein frame as being free from some obstacles related to an appearance of bad singularities. It is an argument for the choice of the Einstein frame as physical.
- We estimated the model parameters of the Starobinsky model in the Palatini formalism in the Jordan frame using astronomical data.

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The authors acknowledge valuable discussion with prof. Andrzej Borowiec, dr Aleksander Stachowski,.