Sasaki-Ricci flow on the Sasaki-Einstein space $T^{1,1}$

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References

Outline

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A $(2n + 1)$-dimensional manifold $M$ is a **contact manifold** if there exists a 1-form $\eta$ (called a contact 1-form) on $M$ such that

$$\eta \wedge (d\eta)^{n-1} \neq 0.$$ 

Associated with a contact form $\eta$ there exists a unique vector field $\xi$ called the **Reeb vector field** defined by the contractions (interior products):

$$i(\xi)\eta = 1,$$

$$i(\xi)d\eta = 0.$$
A simple and direct definition of the Sasakian structures is the following:

A compact Riemannian manifold \((M, g)\) is **Sasakian** if and only if its metric cone \((C(M) \cong \mathbb{R}_+ \times M, \bar{g} = dr^2 + r^2 g)\) is Kähler.

Here \(r \in (0, \infty)\) may be considered as a coordinate on the positive real line \(\mathbb{R}_+\). The Sasakian manifold \((M, g)\) is naturally isometrically embedded into the metric cone via the inclusion \(M = \{r = 1\} = \{1\} \times M \subset C(M)\).

Let us denote by \(L_\xi\) the line subbundle generated by \(\xi\) and let \(\mathcal{D} = \text{Ker} \eta\) be the contact subbundle in \(TM\). Then we have the following decomposition of the tangent bundle \(TM\) of \(M\):

\[
TM = \mathcal{D} \oplus L_\xi.
\]
Sasaki manifold and Sasaki potential (3)

$M$ can be endowed with a contact structure $(\Phi, \xi, \eta)$, where the endomorphism $\Phi$ of the tangent bundle $TM$ is

$$\Phi(X) = \nabla_X \xi,$$

for any smooth vector field $X$ on $M$.

One gets a global 2-form $\Omega^T$ on $M$ coming from the contact 1-form $\eta$, namely

$$\Omega^T = \frac{1}{2} d\eta.$$

We have that $(\mathcal{D}, \Phi|_{\mathcal{D}}, d\eta)$ gives $M$ a transverse Kähler structure with Kähler form $\Omega^T$ defined above and transverse metric $g^T$ given by

$$g^T(X, Y) = d\eta(X, \Phi Y),$$

and related to the Sasaki metric $g$ on $M$ by

$$g = g^T + \eta \otimes \eta.$$
We recall that a Riemannian manifold \((M, g)\) that satisfies the Einstein equation

\[ \text{Ric}_g = \lambda g , \]

for a real constant \(\lambda\) (called Einstein constant), where \(\text{Ric}_g\) stands for the Ricci tensor of the metric \(g\), is said to be an Einstein manifold. Moreover, if the Einstein constant is zero, then the Riemannian space \((M, g)\) is called a Ricci-flat manifold. A Sasaki manifold is said to be a Sasaki-Einstein space if the cone manifold \(C(M)\) of \(M\) is Kähler Ricci-flat (Calabi-Yau). It is clear that a Sasaki-Einstein space is a Riemannian manifold that is both a Sasaki manifold and an Einstein space.

Notice that the transverse metric associated with a Sasaki-Einstein space is Einstein.
Sasaki manifold and Sasaki potential (5)

Every \((2n + 1)\)-dimensional Sasakian manifold is locally generated by a free real-valued function \(K\) of \(2n\) variables, called the Sasaki potential, while every locally Sasaki-Einstein space of dimension \(2n + 1\) is generated by a locally Kähler-Einstein space of dimension \(2n\).

If \(\{U_\alpha\}\) is a foliation chart on \(M\) with \(U_\alpha = I \times V_\alpha\) (where \(I \subset \mathbb{R}\) is an open interval and \(V_\alpha \subset \mathbb{C}^n\)), and \((x, z^1, \ldots, z^n)\) are the local holomorphic coordinates on \(U_\alpha\) (with Reeb vector field \(\xi = \frac{\partial}{\partial x}\) and \(z^1, \ldots, z^n\) are the local holomorphic coordinates on \(V_\alpha\)).
The Sasaki potential [M. Godliński, W. Kopczyński, P. Nurowski, Class. Quantum Grav. 17 (2000) L105-L115]

K on $U_\alpha$ is chosen in such a way that $\xi(K) = 0$ and

$$\eta = dx + i \sum_{j=1}^{n} (K_j dz^j) - i \sum_{\bar{j}=1}^{n} (K_{\bar{j}} d\bar{z}^j),$$

$$d\eta = -2i \sum_{j, \bar{k}=1}^{n} K_{j\bar{k}} dz^j \wedge d\bar{z}^k,$$

$$g = \eta^2 + 2 \sum_{j, \bar{k}=1}^{n} K_{j\bar{k}} dz^j d\bar{z}^k$$

$$\phi = -i \sum_{j=1}^{n} [(\partial_j - iK_j \partial_x) \otimes dz^j] + i \sum_{\bar{j}=1}^{n} (\partial_{\bar{j}} + iK_{\bar{j}} \partial_x) \otimes d\bar{z}^j].$$
We recall that a $r$-form $\alpha$ on $M$ is called \textbf{basic} if

$$\iota_\xi \alpha = 0, \quad \mathcal{L}_\xi \alpha = 0,$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to the vector field $\xi$. In particular a function $\varphi$ is basic if and only if $\xi(\varphi) = 0$. In the system of coordinates $(x, z^1, \ldots, z^n)$ given above, a basic $r$-form of type $(p, q)$, $r = p + q$ has the form

$$\alpha = \alpha_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q},$$

where $\alpha_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q}$ does not depend on $x$. 
Sasaki-Ricci flow (1)

Let $M$ be a smooth manifold equipped with a Sasakian structure $(g, \eta, \xi, \Phi)$. Suppose that we deform the contact form $\eta$ with a basic function $\varphi$ as follows:

$$\tilde{\eta} = \eta + d^c_B \varphi,$$

where $d^c_B = \frac{i}{2}(\bar{\partial}_B - \partial_B)$, $d_B = \partial_B + \bar{\partial}_B$ and $\bar{\partial}_B, \partial_B$ denote the basic Dolbeault operators.

The above deformation implies that other fundamental tensors are also modified:

$$\tilde{\Phi} = \Phi - (\xi \otimes (d^c_B \varphi)) \circ \Phi,$$

$$\tilde{g} = d\tilde{\eta} \circ (\mathbb{1} \otimes \tilde{\Phi}) + \tilde{\eta} \otimes \tilde{\eta},$$

as well as the transverse form:

$$d\tilde{\eta} = d\eta + d_B d^c_B \varphi.$$

It is known that the quadruplet $(\tilde{g}, \tilde{\eta}, \xi, \tilde{\Phi})$ remains a Sasakian structure on $M$. 
Sasaki-Ricci flow (2)

Let \((g(t), \eta(t), \xi, \Phi(t))\) be a flow having initial data \((g(0), \eta(0), \xi, \Phi(0)) = (g, \eta, \xi, \Phi)\), generated by a basic function \(\varphi(t)\) as above and suppose that the basic first Chern class is positive, i.e. \(c^1_B > 0\). Then the **Sasaki-Ricci flow**, also known as **transverse Kähler-Ricci flow** [K. Smoczyk, G. Wang, Y. Zhang, Intern. J. Math. 21 (2010), 951-969; A. Futaki, H. Ono, G. Wang, J. Diff. Geom. 83 (2009), 585-635] is defined by

\[
\frac{\partial g^T}{\partial t} = -Ric^T_{g(t)} + (2n + 2)g^T(t),
\]

where \(Ric^T\) is the transverse Ricci curvature.
Sasaki-Ricci flow (3)

Considering a deformation of the Sasaki structure with a basic function $\varphi$, in local coordinates the Sasaki-Ricci flow can be expressed as a parabolic Monge-Ampère equation

$$\frac{\partial \varphi}{\partial t} = \log \det(g_{j\bar{k}} + \varphi_{j\bar{k}}) - \log(\det g_{j\bar{k}}^T) + (2n + 2)\varphi.$$
Local coordinates on $T^{1,1}$ (1)

$T^{1,1} = S^2 \times S^3$ is one of the most renowned examples of homogeneous Sasaki-Einstein space in dimension five, the standard metric on this manifold being

$$ds^2(T^{1,1}) = \frac{1}{6}(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) + \frac{1}{9}(d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2,$$

where $\theta_i \in [0, \pi)$, $\phi_i \in [0, 2\pi)$, $i = 1, 2$ and $\psi \in [0, 4\pi)$. 
Local coordinates on $T^{1,1}$ (2)

We consider on $T^{1,1}$ a patch of coordinates $(\psi, w^1, w^2)$, where the real coordinates $\psi$ is for the Reeb flow of the Sasaki structure, with

$$\xi = \frac{1}{3} \frac{\partial}{\partial \psi}.$$

$(z^1, z^2)$ are transverse complex coordinates addressing the transverse Kähler structure. As on $T^{1,1}$ the transverse structure are locally isomorphic to a product $S^2 \times S^2$, we choose

$$z^1 = \tan \frac{\theta_1}{2} e^{i\phi_1},$$

$$z^2 = \tan \frac{\theta_2}{2} e^{i\phi_2}.$$
Local coordinates on $T^{1,1}$ (3)

We consider the Sasaki potential

$$K = \frac{1}{3} \sum_{j} \log(1 + z^j \bar{z}^j) - \frac{1}{6} \sum_{j} \log(z^j \bar{z}^j).$$

For the contact form $\eta$ we get

$$\eta = \frac{1}{3} d\psi + i \sum_j \frac{\partial K}{\partial z^j} dz^j - i \sum_{\bar{j}} \frac{\partial K}{\partial \bar{z}^j} d\bar{z}^j$$

$$= \frac{1}{3} d\psi + \frac{1}{3} \sum_j \cos \theta_j \phi_j.$$
Transverse Kähler-Ricci flow on $T^{1,1}$ (1)

In the case of the space $T^{1,1}$ the Ricci flow equation has the form:

$$
\frac{d\varphi}{dt} = \log \left( \varphi_{1\bar{1}}\varphi_{2\bar{2}} - \varphi_{1\bar{2}}\varphi_{2\bar{1}} \right)
+ \cos^4 \frac{\theta_1}{2} \varphi_{2\bar{2}} + \cos^4 \frac{\theta_2}{2} \varphi_{1\bar{1}} + \cos^4 \frac{\theta_1}{2} \cos^4 \frac{\theta_2}{2}

- \log \left( \cos^4 \frac{\theta_1}{2} \cos^4 \frac{\theta_2}{2} \right) + 6\varphi .
$$

Evaluating the derivatives of the basic $\varphi$ we get:

$$
\varphi_{\bar{j}j} = \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_j} = \cos^4 \frac{\theta_j}{2} \frac{\partial^2 \varphi}{\partial \theta_j^2} + \frac{1}{4} \tan^2 \frac{\theta_j}{2} \frac{\partial^2 \varphi}{\partial \phi_j^2} + \frac{1}{2} \cos^2 \frac{\theta_j}{2} \frac{1}{\tan \frac{\theta_j}{2}} \cos \theta_j \frac{\partial \varphi}{\partial \theta_j} ,
$$

with $1 \leq j \leq 2$
Transverse Kähler-Ricci flow on $T^{1,1}$ (2)

$$\phi_{ji} = \frac{\partial^2 \phi}{\partial z^j \partial \bar{z}^l} = \cos^2 \frac{\theta_j}{2} \cos^2 \frac{\theta_l}{2} e^{i(\phi_l - \phi_j)}$$

$$\cdot \left( \frac{\partial^2 \varphi}{\partial \theta_j \partial \theta_l} + \frac{1}{\sin \theta_j \sin \theta_l} \frac{\partial^2 \varphi}{\partial \phi_j \partial \phi_l} - \frac{i}{\sin \theta_j} \frac{\partial^2 \varphi}{\partial \theta_l \partial \phi_j} + \frac{i}{\sin \theta_l} \frac{\partial^2 \varphi}{\partial \phi_l \partial \theta_j} \right)$$

for $i \neq j$. 
We search after particular solutions of the transverse Kähler-Ricci flow equation. We factorize the dependences on the variable $t$ and angle coordinates as follows:

$$\varphi(t, \theta_1, \theta_2, \phi_1, \phi_2) = f(t) \cdot g(\theta_1, \theta_2, \phi_1, \phi_2).$$

The Ricci flow equation is still quite involved and searching for some explicit solutions we shall assume that the dependence on the angles $(\theta_1, \phi_1), (\theta_2, \phi_2)$ of the function $g$ separates:

$$g(\theta_1, \theta_2, \phi_1, \phi_2) = g_1(\theta_1, \phi_1) + g_2(\theta_2, \phi_2).$$

With this simplifying assumption the mixed derivatives $\varphi_{1\bar{2}}$ and $\varphi_{2\bar{1}}$ vanish.
Transverse Kähler-Ricci flow on $T^{1,1}$ (4)

Moreover we look for solutions satisfying the following additional constraints:

$$
\frac{\partial^2 \varphi}{\partial \theta_1^2} + \frac{1}{\sin^2 \theta_1} \frac{\partial^2 \varphi}{\partial \phi_1^2} + \frac{1}{\tan \theta_1} \frac{\partial \varphi}{\partial \theta_1} = c_1 f(t),
$$

$$
\frac{\partial^2 \varphi}{\partial \theta_2^2} + \frac{1}{\sin^2 \theta_2} \frac{\partial^2 \varphi}{\partial \phi_2^2} + \frac{1}{\tan \theta_2} \frac{\partial \varphi}{\partial \theta_2} = c_2 f(t),
$$

where $c_j$ are some arbitrary real constant.

With these assumptions we get that

$$
\varphi_{1\bar{1}} = \cos^4 \frac{\theta_1}{2} c_1 f(t),
$$

$$
\varphi_{2\bar{2}} = \cos^4 \frac{\theta_2}{2} c_2 f(t)
$$
Transverse Kähler-Ricci flow on $T^{1,1}$ (5)

Ricci flow equation reduces to an ordinary differential equation for $f(t)$:

$$\frac{df(t)}{dt} \cdot g(\theta_1, \theta_2, \phi_1, \phi_2) = \log \left[ f^2(t)(c_1 c_2) + f(t)(c_1 + c_2) + 1 \right]$$

$$+ 6f(t) \cdot g(\theta_1, \theta_2, \phi_1, \phi_2).$$

We search for a solution of the form:

$$g(\theta_1, \theta_2, \phi_1, \phi_2) = \frac{1}{2} d_1 \phi_1^2 + h_1(\theta_1) + \frac{1}{2} d_2 \phi_2^2 + h_2(\theta_2),$$

where $d_j$ are some arbitrary real constants.
Functions $h_j$ are

\[ h_1(\theta_1) = e_1 \log u_1 - \frac{d_1}{2} (\log u_1)^2 - c_1 \log \sin \theta_1 , \]

\[ h_2(\theta_2) = e_2 \log u_2 - \frac{d_2}{2} (\log u_2)^2 - c_2 \log \sin \theta_2 , \]

where

\[ u_j = \frac{\sin \theta_j}{1 + \cos \theta_j} , \quad j = 1, 2 . \]

and $d_j, e_j$ are other arbitrary real constants.
Transverse Kähler-Ricci flow on $T^{1,1}$ (7)

The simplest solution is that involving only the constants $e_j \neq 0$ and the rest of the constants is zero. In that case we can state the following proposition:

**Proposition**

*Any metric of the form with arbitrary real constants $e_j$, $j = 1, 2$*

\[
\tilde{g} = \frac{1}{9} \left( d\psi + \sum_j (\cos \theta_j + \frac{e_j}{2}) d\phi_j \right)^2 + \frac{1}{6} \sum_j \left( d\theta_j^2 + \sin^2 \theta_j d\phi_j^2 \right)
\]

represents a deformation of the canonical metric on $T^{1,1}$. The deformed contact structure remains Sasaki-Einstein with the contact form

\[
\tilde{\eta} = \eta + \frac{1}{6} \sum_j e_j \, d\phi_j = \frac{1}{3} d\psi + \frac{1}{3} \sum_j \cos \theta_j d\phi_j + \frac{1}{6} \sum_j e_j \, d\phi_j.
\]
Transverse Kähler-Ricci flow on $T^{1,1}$ (8)

A more involved deformation can be obtained assuming that the constants $d_j \neq 0$. In this case we get the following deformation of the Sasaki structure:

**Proposition**

*Any metric of the form with arbitrary real constants $d_j$, $j = 1, 2*\

\[
\tilde{g} = \frac{1}{9} \left( d\psi + \sum_j \left( \cos \theta_j + \frac{d_j}{2} \log \tan \frac{\theta_j}{2} \right) d\phi_j \right.
\]

\[
+ \frac{1}{2} \sum_j d_j \frac{\phi_j}{\sin \theta_j} d\theta_j \right)^2 + \frac{1}{6} \sum_j \left( d\theta_j^2 + \sin^2 \theta_j d\phi_j^2 \right)
\]

represents a deformation of the canonical metric on $T^{1,1}$. 
The deformed contact structure remains Sasaki-Einstein with the contact form

\[ \tilde{\eta} = \frac{1}{3} d\psi + \frac{1}{3} \sum_j \cos \theta_j d\phi_j + \frac{1}{2} \sum_j d_j \frac{\phi_j}{\sin \theta_j} d\theta_j \]

\[ + \frac{1}{2} \sum_j d_j \log \tan \frac{\theta_j}{2} d\phi_j. \]
Let us remark that in both deformations considered above the transverse metric remains unaltered. For \( c_1 = c_2 = 0 \) the function \( f(t) \) satisfying the Ricci flow equation has a very simple solution with the initial condition \( f(0) = 0 \):

\[
f(t) = e^{6t} - 1.
\]
Transverse Kähler-Ricci flow on $T^{1,1}$ (11)

To summarize we have the following outcome:

**Corollary**

The families of potential basic functions

$$\varphi_t = (e^{6t} - 1) \sum_j e_j \log z^j \bar{z}^j,$$

and

$$\varphi_t = (e^{6t} - 1) \left[\sum_j d_j \log^2 z^j + \frac{1}{2} \sum_j d_j \log z^j \log z^j \bar{z}^j \right. \left. - \frac{1}{4} \sum_j d_j \log^2 z^j \bar{z}^j \right],$$

stand as solutions of the transverse Kähler-Ricci flow equation on the manifold $T^{1,1}$. 
Transverse Kähler-Ricci flow on $T^{1,1}$ (12)

Finally, let us consider deformations of the Sasaki structures involving the constants $c_j \neq 0$. In this case we have a modification of the transverse metric as follows:

**Proposition**

The deformed contact structure with the contact form

$$\tilde{\eta} = \eta + \sum_j c_j \cos \theta_j d\phi_j = \frac{1}{3} (d\psi + (1 - 3c_j) \cos \theta_j d\phi_j).$$

remains Sasaki with the metric

$$\tilde{g} = \frac{1}{9} \left[ d\psi + \sum_j (1 - 3c_j) \cos \theta_j d\phi_j \right]^2 + \frac{1}{6} \sum_j (1 + 3c_j)(d\theta_j^2 + \sin^2 \theta_j d\phi_j^2).$$
Outlook

- Killing forms on deformed manifolds under Sasaki-Ricci flow
- Integrals of motion on deformed Sasaki-Einstein spaces
- Sasaki-Ricci flow on $Y^{p,q}$
- Sasaki-Ricci flow on 3-Sasakian manifolds
Appendix (1)

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z^j} dz^j,$$

$$\bar{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}^j} d\bar{z}^j.$$

$$Ric^T(X, Y) = Ric(X, Y) + 2g^T(X, Y).$$

Let $\rho^T = Ric^T(\Phi \cdot, \cdot)$ and $\rho = Ric(\Phi \cdot, \cdot)$.

$\rho^T$ is called the **transverse Ricci form**.

$$\rho^T = \rho + 2d\eta.$$
Appendix (2)

Transverse Einstein metric

\[ Ric^T = cg^T. \]

\( \rho^T \) is a closed basic form and its basic cohomology class \([\rho^T]_B = c_B^1\) is the basic first Chern class.

\( c_B^1 \) is called positive (respectively, negative, null) if it contain a positive (respectively, negative, null) representation

\[ c_B^1 = k[d\eta]_B \]

where \( k = +1, -1, 0. \)