3-CATEGORIES, 3-GROUPS, AND UNIFICATION OF GRAVITY AND MATTER

Marko Vojinović, in collaboration with Tijana Radenković

Institute of Physics, Univesrity of Belgrade, Serbia

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INTRODUCTION

The line of development that leads to higher gauge theories:

- Within the LQG framework, the covariant quantization program is based on a "path integral on a lattice" idea, starting from the classical action for GR in the form of a BF theory with a simplicity constraint, known as the Plebanski action. The lattice quantization gives rise to spinfoam models.

- Since the Plebanski action does not contain the tetrad fields in its topological BF sector, the spinfoam models are hard to couple to matter fields. This problem is solved by passing to the $2BF$ action, which is a categorical generalization of the BF theory. The matter fields can now be coupled, albeit in an ad hoc way (as in the Standard Model). The $2BF$ action was a first physically relevant example of a higher gauge theory, and its lattice quantization was termed "spincube model".

- Yet another categorical generalization to the new $3BF$ action manages to combine gravity, gauge fields and matter fields in a unified geometric and algebraic way, paving the way to a "categorical unification" of all fields.

CATEGORY THEORY AND 3-GROUPS

A flash introduction to the category theory "ladder":

- a category $\mathcal{C} = (Obj, Mor)$ is a structure which has objects and morphisms between them,

$$X, Y, Z, \dots \in Obj\,, \qquad f, g, h, \dots \in Mor\,,$$

$$f : X \to Y, \quad g : Z \to X, \quad h : X \to Y, \dots$$

  such that certain rules are respected, like the associativity of morphism composition, etc.

- a 2-category $\mathcal{C}_2 = (Obj, Mor_1, Mor_2)$ is a structure which has objects, morphisms between them, and morphisms between morphisms, called 2-morphisms,

$$X, Y, Z, \dots \in Obj\,, \qquad f, g, h, \dots \in Mor_1\,, \qquad \alpha, \beta, \dots \in Mor_2\,,$$

$$f : X \to Y, \quad g : Z \to X, \quad h : X \to Y, \dots \qquad \alpha : f \to h\,, \dots$$

  such that similar rules as above are respected.

CATEGORY THEORY AND 3-GROUPS

- a 3-category $\mathcal{C}_3 = (Obj, Mor_1, Mor_2, Mor_3)$ additionally has morphisms between 2-morphisms, called 3-morphisms,

$$\Theta, \Phi, \dots \in Mor_3 \,, \qquad \Theta : \alpha \to \beta \,, \dots$$

  again with a certain set of axioms about compositions of various n-morphisms.

- one can further generalize these structures to introduce 4-categories, n-categories, ∞-categories, etc.

The algebraic structure of a group is a special case of a category:

- a group is a category with only one object, while all morphisms are invertible;

- a 2-group is a 2-category with only one object, while all 1-morphisms and 2-morphisms are invertible;

- a 3-group is a 3-category with only one object, while all 1-morphisms, 2-morphisms and 3-morphisms are invertible.

LIE 3-GROUPS

A more practical way to talk about 3-group — a 2-crossed module:

$$L \xrightarrow{\delta} H \xrightarrow{\partial} G \,,$$

- for our purposes L, H and G are ordinary Lie groups,
- there are two "boundary homomorphisms" δ and ∂,
- there is a defined action $\triangleright$ of G onto G, H and L,

$$\triangleright : G \times G \to G \,, \qquad \triangleright : G \times H \to H \,, \qquad \triangleright : G \times L \to L \,,$$

- there is a bracket operation over H to L,

$$\{ \quad , \quad \} : H \times H \to L \,,$$

- and certain axioms are assumed to hold true among all these maps.

LIE 3-GROUPS

The axioms of a 2-crossed module $L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G$:

$$g \rhd \partial h = \partial(g \rhd h)\,,$$

$$g \rhd \delta l = \delta(g \rhd l)\,,$$

$$g \rhd g_0 = g\, g_0\, g^{-1}\,,$$

$$g \rhd \{h_1, h_2\} = \{g \rhd h_1, g \rhd h_2\}\,,$$

$$\partial\delta = 1_G\,,$$

$$\delta\,\{h_1, h_2\} = h_1 h_2 h_1^{-1} (\partial h_1) \rhd h_2^{-1}\,,$$

$$\{\delta l_1, \delta l_2\} = l_1 l_2 l_1^{-1} l_2^{-1}\,,$$

$$\{h_1 h_2, h_3\} = \{h_1, h_2 h_3 h_2^{-1}\}\, \partial h_1 \rhd \{h_2, h_3\}\,,$$

$$\{\delta l, h\}\,\{h, \delta l\} = l(\partial h \rhd l^{-1})\,.$$

. . . for all $g \in G$, $h \in H$ and $l \in L$. . . :-)

LIE 3-GROUPS

A Lie 3-group has a corresponding Lie 3-algebra, i.e. a differential 2-crossed module:

$$\mathfrak{l} \xrightarrow{\delta} \mathfrak{h} \xrightarrow{\partial} \mathfrak{g} \,,$$

- where $\mathfrak{l}$, $\mathfrak{h}$, $\mathfrak{g}$ are Lie algebras of L, H, G,
- the maps δ, ∂, $\triangleright$ and $\{\ \ ,\ \ \}$ are inherited from the 3-group,
- "corresponding" axioms apply.

In addition to all this, Lie algebras have their own Lie structure:

- generators,

$$T_A \in \mathfrak{l} \,, \qquad t_a \in \mathfrak{h} \,, \qquad \tau_\alpha \in \mathfrak{g}$$

- structure constants,

$$[T_A, T_B] = f_{AB}{}^C T_C \,, \qquad [t_a, t_b] = f_{ab}{}^c t_c \,, \qquad [\tau_\alpha, \tau_\beta] = f_{\alpha\beta}{}^\gamma \tau_\gamma \,,$$

- and symmetric bilinear invariant Killing forms,

$$\langle T_A, T_B \rangle_{\mathfrak{l}} = g_{AB} \,, \qquad \langle t_a, t_b \rangle_{\mathfrak{h}} = g_{ab} \,, \qquad \langle \tau_\alpha, \tau_\beta \rangle_{\mathfrak{g}} = g_{\alpha\beta} \,.$$

LIE 3-GROUPS

The main purpose of all this structure is to generalize the notion of parallel transport from curves to surfaces to volumes:

- Given a 4-dimensional manifold $\mathcal{M}$, define a 3-connection (α, β, γ) as a triple of 3-algebra-valued differential forms,

$$\begin{array}{rcll} \alpha & = & \alpha^{\alpha}{}_{\mu}(x)\,\tau_{\alpha}\,\mathbf{d}x^{\mu} & \in \Lambda^1(\mathcal{M},\mathfrak{g})\,, \\ \beta & = & \frac{1}{2}\beta^{a}{}_{\mu\nu}(x)\,t_a\,\mathbf{d}x^{\mu}\wedge\mathbf{d}x^{\nu} & \in \Lambda^2(\mathcal{M},\mathfrak{h})\,, \\ \gamma & = & \frac{1}{3!}\gamma^{A}{}_{\mu\nu\rho}(x)\,T_A\,\mathbf{d}x^{\mu}\wedge\mathbf{d}x^{\nu}\wedge\mathbf{d}x^{\rho} & \in \Lambda^3(\mathcal{M},\mathfrak{l})\,. \end{array}$$

- Then introduce the line, surface and volume holonomies,

$$g = \mathcal{P}\exp\int_{\mathcal{C}_1}\alpha\,, \qquad h = \mathcal{P}\exp\int_{\mathcal{S}_2}\beta\,, \qquad l = \mathcal{P}\exp\int_{\mathcal{V}_3}\gamma\,,$$

- and corresponding curvature forms,

$$\begin{array}{rcl} \mathcal{F} & = & \mathbf{d}\alpha + \alpha\wedge\alpha - \partial\beta\,, \\ \mathcal{G} & = & \mathbf{d}\beta + \alpha\wedge^{\triangleright}\beta - \delta\gamma\,, \\ \mathcal{H} & = & \mathbf{d}\gamma + \alpha\wedge^{\triangleright}\gamma - \{\beta\wedge\beta\}\,. \end{array}$$

HIGHER GAUGE THEORIES

At this point one can construct the so-called $3BF$ theory, with the action:

$$S_{3BF} = \int_{\mathcal{M}} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}} \,.$$

- $3BF$ theory is a topological gauge theory,
- it is based on the 3-group structure,
- it is a generalization of an ordinary BF theory for a given Lie group G,

The physical interpretation of the Lagrange multipliers C and D:

- the $\mathfrak{h}$-valued 1-form C can be interpreted as the tetrad field, if $H = \mathbb{R}^4$ is the spacetime translation group:

$$C \to e = e^a{}_\mu(x)\, t_a \, \mathbf{d}x^\mu \,,$$

- the $\mathfrak{l}$-valued 0-form D can be interpreted as the set of real-valued matter fields, given some Lie group L:

$$D \to \phi = \phi^A(x)\, T_A \,.$$

HIGHER GAUGE THEORIES

How to choose the 3-group? The simplest example — the (trivial) Standard Model 3-group:

$$G = SO(3,1) \times SU(3) \times SU(2) \times U(1)\,, \qquad H = \mathbb{R}^4\,, \qquad L \text{ to be discussed.}$$

- boundary maps are trivial — for all $l \in L$ and $\vec{v} \in H$, we define

$$\delta l = 1_H = 0\,, \qquad \partial \vec{v} = 1_G\,,$$

- the bracket is trivial — for all $\vec{u}, \vec{v} \in H$, we define

$$\{\vec{u}, \vec{v}\} = 1_L\,,$$

- the action $\triangleright$ of G on itself is via the adjoint representation, the action on H is via vector representation for the $SO(3,1)$ sector and via trivial representation for the $SU(3) \times SU(2) \times U(1)$ sector.

- the action of G on L is nontrivial and depends on the choice of L (to be discussed).

One can verify that all axioms of a 3-group are satisfied.

HIGHER GAUGE THEORIES

How to choose L? Study the $3BF$ action:

$$S_{3BF} = \int_{\mathcal{M}} B^\alpha \wedge \mathcal{F}^\beta g_{\alpha\beta} + e^a \wedge \mathcal{G}^b g_{ab} + \phi^A \mathcal{H}^B g_{AB} \,.$$

- The indices α of G split according to its structure, as $\alpha = (ab\,,\, i)$, giving the connection and curvature

$$\alpha = \omega^{ab} J_{ab} + A^i \tau_i \,, \qquad \mathcal{F} = R^{ab} J_{ab} + F^i \tau_i \,.$$

- The vectorial action of $SO(3,1)$ on $H = \mathbb{R}^4$ implies the Minkowski signature of the bilinear invariant on $\mathbb{R}^4$, so that

$$g_{ab} = \eta_{ab} \equiv \mathrm{diag}(-1,+1,+1,+1) \,.$$

- Given that $\phi = \phi^A T_A$, we have one real-valued field $\phi^A(x)$ for each generator $T_A \in \mathfrak{l}$.

THE STANDARD MODEL

How many real-valued field components do we have in the matter sector of the Standard Model? The fermion sector gives us:

$$\left.\begin{matrix} \begin{pmatrix}\nu_e \\ e^-\end{pmatrix}_L & \begin{pmatrix}u_r \\ d_r\end{pmatrix}_L & \begin{pmatrix}u_g \\ d_g\end{pmatrix}_L & \begin{pmatrix}u_b \\ d_b\end{pmatrix}_L \\ \\ (\nu_e)_R & (u_r)_R & (u_g)_R & (u_b)_R \\ \\ (e^-)_R & (d_r)_R & (d_g)_R & (d_b)_R \end{matrix}\right\} = 16\ \frac{\text{spinors}}{\text{family}} \times$$

$$\times 3 \text{ families } \times 4\ \frac{\text{real-valued fields}}{\text{spinor}} = 192 \text{ real-valued fields } \phi^A\,.$$

The Higgs sector gives us:

$$\left.\begin{pmatrix}\phi^+ \\ \phi_0\end{pmatrix}\right\} = 2 \text{ complex scalar fields } = 4 \text{ real-valued fields } \phi^A\,.$$

This suggests the structure for L in the form:

$$L = L_{\text{fermion}} \times L_{\text{Higgs}}\,, \qquad \dim L_{\text{fermion}} = 192\,, \qquad \dim L_{\text{Higgs}} = 4\,.$$

THE STANDARD MODEL

The action $\triangleright : G \times L \to L$ specifies the transformation properties of each real-valued field ϕ^A with respect to Lorentz and internal symmetries.

- For example, in

$$\begin{pmatrix} u_b \\ d_b \end{pmatrix}_L$$

  the action

$$g \triangleright u_b \,, \qquad g \in SO(3,1) \times SU(3) \times SU(2) \times U(1)\,,$$

  encodes that u_b consists of 4 real-valued fields which transform as:

  - a left-handed spinor wrt. $SO(3,1)$,
  - as a “blue” component of the fundamental representation of $SU(3)$,
  - and as “isospin $+\frac{1}{2}$” of the left doublet wrt. $SU(2) \times U(1)$.

- Moreover, G acts in the same way across families, suggesting the structure

$$L_{\text{fermion}} = L_{\text{1st family}} \times L_{\text{2nd family}} \times L_{\text{3rd family}}\,, \qquad \dim L_{k\text{-th family}} = 64\,.$$

THE STANDARD MODEL

A realisation that reproduces the Standard Model:

- Choice of the 3-group:

$$L = \mathbb{R}^4(\mathbb{C}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \,,$$

$$G = SO(3,1) \times SU(3) \times SU(2) \times U(1) \,, \qquad H = \mathbb{R}^4 \,,$$

  where $\mathbb{G}$ is the Grassmann algebra.

- The constrained $3BF$ action:

$$S = \int B_{\hat{\alpha}} \wedge \mathcal{F}^{\hat{\alpha}} + e_{\hat{a}} \wedge \mathcal{G}^{\hat{a}} + D_{\hat{A}} \wedge \mathcal{H}^{\hat{A}}$$

$$+ \left(B_{\hat{\alpha}} - C_{\hat{\alpha}}{}^{\hat{\beta}} M_{cd\hat{\beta}} e^c \wedge e^d \right) \wedge \lambda^{\hat{\alpha}} - \left(\gamma_{\hat{A}} - e^a \wedge e^b \wedge e^c C_{\hat{A}}{}^{\hat{B}} M_{abc\hat{B}} \right) \wedge \lambda^{\hat{A}} - 4\pi i \, l_p^2 \, \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c D_{\hat{A}} T^{d\hat{A}}{}_{\hat{B}} D^{\hat{B}}$$

$$+ \zeta^{ab}{}_{\hat{\alpha}} \wedge \left(M_{ab}{}^{\hat{\alpha}} \varepsilon^{cdef} e_c \wedge e_d \wedge e_e \wedge e_f - F^{\hat{\alpha}} \wedge e_c \wedge e_d \right) + \zeta^{ab}{}_{\hat{A}} \wedge \left(M_{abc}{}^{\hat{A}} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - F^{\hat{A}} \wedge e_a \wedge e_b \right)$$

$$- \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \, \left(M_{\hat{A}\hat{B}} D^{\hat{A}} D^{\hat{B}} + Y_{\hat{A}\hat{B}\hat{C}} D^{\hat{A}} D^{\hat{B}} D^{\hat{C}} + L_{\hat{A}\hat{B}\hat{C}\hat{D}} D^{\hat{A}} D^{\hat{B}} D^{\hat{C}} D^{\hat{D}} \right) \,.$$

CONCLUSIONS

- Higher gauge theory represents a formalism where gravity, gauge fields, fermions and Higgs are treated on an equal footing.
- The underlying algebraic structure of a 3-group classifies all fundamental fields by specifying groups L, H, G and their maps $\delta, \partial, \triangleright, \{\ ,\ \}$.
- This structure has natural geometrical interpretation of parallel transport along a curve, a surface, and a volume.
- The gauge group L specifies the complete matter sector of the Standard Model if one chooses
$$L = \mathbb{R}^4(\mathbb{C}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G})\,.$$
- The action $\triangleright$ of G on L specifies the transformation properties of matter fields.
- Nontrivial choices of the 3-group structure may provide new avenues for research on unification of all fields.

THANK YOU!